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On the coadjoint representation of $\mathbb{Z}_2$-contractions of reductive Lie algebras ☆

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Abstract

We study the coadjoint representation of contractions of reductive Lie algebras associated with symmetric decompositions. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a symmetric decomposition of a reductive Lie algebra $\mathfrak{g}$. Then the semi-direct product of $\mathfrak{g}_0$ and the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is a contraction of $\mathfrak{g}$. We conjecture that these contractions have many properties in common with reductive Lie algebras. In particular, it is proved that in many cases the algebra of invariants is polynomial. We also discuss the so-called "codim-2 property" for coadjoint representations and its relationship with the structure of algebra of invariants.

Keywords: Semisimple Lie algebra; Coadjoint representation; Symmetric pairs; Algebra of invariants

Contents

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0. Introduction

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Let $\mathfrak{g}$ be a reductive algebraic Lie algebra. Classical results of Kostant [7] give a fairly complete invariant-theoretic picture of the (co)adjoint representation of $\mathfrak{g}$. Let $\sigma \in \mathrm{Aut}(\mathfrak{g})$ be an involution and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the corresponding $\mathbb{Z}_2$-grading. Associated to this decomposition, there is a non-reductive Lie algebra $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, the semi-direct product of the Lie algebra $\mathfrak{g}_0$ and $\mathfrak{g}_0$-module $\mathfrak{g}_1$. Let K denote a connected group with Lie algebra $\mathfrak{k}$. A remarkable property of the Lie algebra contraction $\mathfrak{g} \rightsquigarrow \mathfrak{k}$ is that it preserves the transcendence degree of the algebras of invariants for both adjoint and coadjoint representations of $\mathfrak{k}$; i.e., $\operatorname{trdeg} \Bbbk[\mathfrak{k}]^K = \operatorname{trdeg} \Bbbk[\mathfrak{k}^*]^K = \operatorname{rk} \mathfrak{g}$. The latter equality also shows that $\operatorname{ind} \mathfrak{k} = \operatorname{rk} \mathfrak{g}$. In [13], we proved that many good properties of Kostant's picture for $(\mathfrak{g}, \mathrm{ad})$ carry over to $(\mathfrak{k}, \mathrm{ad})$. In particular, $\Bbbk[\mathfrak{k}]^K$ is a polynomial algebra and the quotient mapping $\pi_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{k}/\!\!/K = \operatorname{Spec}(\Bbbk[\mathfrak{k}]^K)$ is equidimensional. The goal of this article is to study the invariants of $(\mathfrak{k}, \mathrm{ad}^*)$. Motivated by several examples, we come up with the following

0.1. Conjecture. *The algebra of invariants of* $(\mathfrak{k}, \mathrm{ad}^*)$ *is polynomial and the quotient mapping* $\pi_{\mathfrak{k}^*} : \mathfrak{k}^* \to \mathfrak{k}^*/\!\!/K = \operatorname{Spec}(\Bbbk[\mathfrak{k}^*]^K)$ *is equidimensional.*

If $\mathfrak{s}$ is reductive, $\mathfrak{g} = \mathfrak{s} \dotplus \mathfrak{s}$, and $\sigma \in \mathrm{Aut}(\mathfrak{g})$ is the permutation, then $\mathfrak{k} = \mathfrak{s} \ltimes \mathfrak{s}$ is a so-called *Takiff* Lie algebra. Here $\mathrm{ad} \simeq \mathrm{ad}^*$ and the validity of the conjecture follows from results of Takiff [21] and Geoffriau [4] (see also [13]). Therefore one can concentrate on the case in which $\mathfrak{g}$ is simple, where the adjoint and coadjoint representations of $\mathfrak{k}$ are different. It is not hard to prove that if $\mathfrak{g}_1$ contains a Cartan subalgebra of $\mathfrak{g}$ (the "maximal rank" case), then the conjecture is true. More generally, we prove Conjecture 0.1 (and some stronger assertions) for the $\mathcal{N}$*-regular* $\mathbb{Z}_2$-gradings, i.e., if $\mathfrak{g}_1$ contains a regular nilpotent element of $\mathfrak{g}$ (see Section 5). There are also several cases, where we can prove only "half" of the conjecture, i.e., the fact that $\Bbbk[\mathfrak{k}^*]^K$ is polynomial (Section 4). Our proofs of polynomiality in Section 4 make use of some general results on coadjoint representations. We show that $(\mathfrak{k}, \mathrm{ad}^*)$ has a so-called *codim-2 property*, i.e., the set of non-regular elements of $\mathfrak{k}^*$ is of codimension $\geqslant 2$ (Theorem 3.3). This property implies, in turn, that if $l = \operatorname{ind} \mathfrak{k}$ and $F_1, \dots, F_l \in \Bbbk[\mathfrak{k}^*]^K$ are homogeneous and algebraically independent, then

$$\sum_{i=1}^{l} \deg F_i \geqslant (\dim \mathfrak{k} + l)/2. \tag{0.2}$$

Furthermore, if the equality holds, then $F_1, \dots, F_l$ freely generate $\Bbbk[\mathfrak{k}^*]^K$ (see Theorem 1.2). That is, the polynomiality follows if one could find algebraically independent K-invariants with "sufficiently small" degrees. To this end, we use the method of $\mathbb{Z}_2$*-degeneration* of G-invariants in $\Bbbk[\mathfrak{g}]$. Namely, the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ determines the natural bi-grading of $\Bbbk[\mathfrak{g}]$. For a homogeneous $f \in \Bbbk[\mathfrak{g}]^G$, let $f^\bullet$ be the bi-homogeneous component of f of highest degree with respect to $\mathfrak{g}_1$. Then regarded as function on $\mathfrak{k}^*$, $f^\bullet$ is K-invariant (Proposition 3.1). Notice that $\deg f^\bullet = \deg f$. It is also known that if $f_1, \dots, f_l \in \Bbbk[\mathfrak{g}]^G$ are basic invariants, then

$\sum_{i=1}^{l} \deg f_i = (\dim \mathfrak{g} + l)/2$. Hence it suffices to find a set of basic G-invariants $f_1, \dots, f_l$ such that $f_1^\bullet, \dots, f_l^\bullet$ are algebraically independent. In this situation, we say that $f_1, \dots, f_l$ form a *good generating system* for $(\mathfrak{g}, \mathfrak{g}_0)$. Then the functions $\{f^\bullet \mid f \in \Bbbk[\mathfrak{g}]^G\}$ form the whole algebra $\Bbbk[\mathfrak{k}^*]^K$. However, this is not always the case (see Remark 4.3). Therefore the proof of polynomiality for some symmetric pairs requires different ideas.

Inequality (0.2) holds for any Lie algebra with the codim-2 property. But $\Bbbk[\mathfrak{k}^*]^K$ is bi-graded (Theorem 2.3), and we also prove a bi-graded refinement of that inequality (see Theorem 3.6). In the last section, we gather the available information on the bi-degrees of basic invariants for $\Bbbk[\mathfrak{k}]^K$ and $\Bbbk[\mathfrak{k}^*]^K$.

All Lie algebras are assumed to be algebraic. Algebraic groups are denoted by capital Latin letters. Corresponding Lie algebras are denoted by the lowercase Gothic letters.

If an algebraic group Q acts on an irreducible affine variety X, then $\Bbbk[X]^Q$ is the algebra of Q-invariant regular functions on X and $\Bbbk(X)^Q$ is the field of Q-invariant rational functions. If $\Bbbk[X]^Q$ is finitely generated, then $X/\!\!/Q := \operatorname{Spec} \Bbbk[X]^Q$, and the *quotient morphism* $\pi_X : X \to X/\!\!/Q$ is the mapping associated with the embedding $\Bbbk[X]^Q \hookrightarrow \Bbbk[X]$. If $\Bbbk[X]^Q$ is graded polynomial, then the elements of any set of algebraically independent homogeneous generators will be referred to as *basic invariants*. Occasionally, we write $Inv(\mathfrak{q}, \mathrm{ad})$ and $Inv(\mathfrak{q}, \mathrm{ad}^*)$ for the algebras of invariants of the adjoint and coadjoint representations of $\mathfrak{q} = \operatorname{Lie} Q$, respectively. If V is a Q-module, then $\mathfrak{q}_v$ is the stabiliser of $v \in V$ in $\mathfrak{q}$. For the adjoint representation of $\mathfrak{q}$, the stabiliser of $x \in \mathfrak{q}$ is also denoted by $\mathfrak{z}_{\mathfrak{q}}(x)$. A direct sum of Lie algebras is denoted by '$\dotplus$.'

Given an irreducible variety Y, an open subset $\Omega \subset Y$ is said to be *big* if $Y \setminus \Omega$ contains no divisors.

$[n] = \{1, 2, \dots, n\}$; $\lfloor x \rfloor$ is the least integer not exceeding x.

1. The codim-2 property for coadjoint representations

Let Q be a connected algebraic group with Lie algebra $\mathfrak{q}$. Let $\mathfrak{q}^*_{\mathrm{reg}}$ denote the set of all *Q-regular* elements of $\mathfrak{q}^*$. That is,

$$\mathfrak{q}^*_{\mathrm{reg}} = \{\xi \in \mathfrak{q}^* \mid \dim(Q \cdot \xi) \geqslant \dim(Q \cdot \eta) \text{ for all } \eta \in \mathfrak{q}^*\}.$$

As is well known, $\mathfrak{q}^*_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{q}^*$.

1.1. Definition. We say that the coadjoint representation of $\mathfrak{q}$ has the *codim-2 property* if $\operatorname{codim}(\mathfrak{q}^* \setminus \mathfrak{q}^*_{\mathrm{reg}}) \geqslant 2$, i.e., $\mathfrak{q}^*_{\mathrm{reg}}$ is big.

Example. If $\mathfrak{g}$ is reductive, then $\mathrm{ad} \simeq \mathrm{ad}^*$ and $\operatorname{codim}(\mathfrak{g} \setminus \mathfrak{g}_{\mathrm{reg}}) = 3$. Hence the coadjoint representation of a reductive Lie algebra has the codim-2 property.

If $\xi \in \mathfrak{q}^*_{\mathrm{reg}}$, then $\dim \mathfrak{q}_\xi$ is called the *index* of $\mathfrak{q}$, denoted $\operatorname{ind} \mathfrak{q}$. Recall that each orbit $Q \cdot \xi$ is a symplectic variety and hence $\dim(Q \cdot \xi)$ is even. By Rosenlicht's theorem, $\operatorname{trdeg} \Bbbk(\mathfrak{q}^*)^Q = \operatorname{ind} \mathfrak{q}$. It follows that if $f_1, \dots, f_r \in \Bbbk[\mathfrak{q}^*]^Q$ are algebraically independent, then $r \leqslant \operatorname{ind} \mathfrak{q}$.

Importance of the codim-2 property is explained by the following result, which makes use of some ideas of [10, Theorem 3.1] (cf. also [14, Theorem 1.2]).

1.2. Theorem. *Suppose that* $(\mathfrak{q}, \mathrm{ad}^*)$ *has the codim-2 property and* $\operatorname{trdeg} \Bbbk[\mathfrak{q}^*]^Q = \operatorname{ind} \mathfrak{q}$. *Set* $l = \operatorname{ind} \mathfrak{q}$. *Let* $f_1, \dots, f_l \in \Bbbk[\mathfrak{q}^*]^Q$ *be arbitrary homogeneous algebraically independent polynomials. Then*

(i) $\sum_{i=1}^{l} \deg f_i \geqslant (\dim \mathfrak{q} + \operatorname{ind} \mathfrak{q})/2$;
(ii) *If* $\sum_{i=1}^{l} \deg f_i = (\dim \mathfrak{q} + \operatorname{ind} \mathfrak{q})/2$, *then* $\Bbbk[\mathfrak{q}^*]^Q$ *is freely generated by* $f_1, \dots, f_l$ *and* $\xi \in \mathfrak{q}^*_{\mathrm{reg}}$ *if and only if* $(df_1)_\xi, \dots, (df_l)_\xi$ *are linearly independent.*

Proof. Recall that $\mathcal{S}(\mathfrak{q}) = \Bbbk[\mathfrak{q}^*]$ is a Poisson algebra, and the symplectic leaves in $\mathfrak{q}^*$ are precisely the coadjoint orbits of Q. Let $\{,\}$ denote the Poisson bracket in $\Bbbk[\mathfrak{q}^*]$. Then $\Bbbk[\mathfrak{q}^*]^Q$ is the centre of $(\Bbbk[\mathfrak{q}^*], \{,\})$.

Let π denote the Poisson tensor (bi-vector) on $\mathfrak{q}^*$. If $T(\mathfrak{q}^*)$ is the tangent bundle of $\mathfrak{q}^*$, then π is a section of $\wedge^2 T(\mathfrak{q}^*)$. By definition, if $f_1, f_2 \in \mathcal{S}(\mathfrak{q})$, then $\pi(df_1, df_2) = \{f_1, f_2\}$. In particular, if $x, y \in \mathfrak{q}$, then $\pi(dx, dy) = [x, y]$. We regard π as an element of the graded skew-symmetric algebra of polynomial vector fields on $\mathfrak{q}^*$. Set $n = \dim \mathfrak{q}$ and $l = \operatorname{ind} \mathfrak{q}$. Let $\operatorname{rk} \pi_\xi$ denote the rank of the bi-vector π at $\xi \in \mathfrak{q}^*$. It is easily seen that $\operatorname{rk} \pi_\xi = \dim(Q \cdot \xi)$. Therefore

$$\left\{\xi \in \mathfrak{q}^* \;\middle|\; \operatorname{rk} \pi_\xi < n - l\right\} = \mathfrak{q}^* \setminus \mathfrak{q}^*_{\mathrm{reg}}.$$

It follows from the definition of index that

$$\mathfrak{V}_1 := \underbrace{\pi \wedge \pi \wedge \cdots \wedge \pi}_{(n-l)/2}$$

is the maximal nonzero exterior power of π. It is an $(n-l)$-vector field on $\mathfrak{q}^*$ of degree $(n-l)/2$ and $(\mathfrak{V}_1)_\xi = 0$ if and only if $\xi \notin \mathfrak{q}^*_{\mathrm{reg}}$.

On the other hand, given algebraically independent polynomials $f_1, \dots, f_l \in \Bbbk[\mathfrak{q}^*]^Q$, we get the nonzero differential l-form, $df_1 \wedge \cdots \wedge df_l$, on $\mathfrak{q}^*$. Let $x_1, \dots, x_n$ be a basis for $\mathfrak{q}$. For an l-form $\mathfrak{F}$ on $\mathfrak{q}^*$, let $\mathfrak{F}^\diamond$ denote the $(n-l)$-vector field defined by the formula

$$\mathfrak{F}^\diamond(\Psi_1, \dots, \Psi_{n-l}) = \frac{\mathfrak{F} \wedge \Psi_1 \wedge \cdots \wedge \Psi_{n-l}}{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}$$

for arbitrary differential 1-forms Ψ_i. It is easily seen that the operation '$\diamond$' does not affect the degree. That is, if $\mathfrak{F}$ is homogeneous, then so is $\mathfrak{F}^\diamond$, and $\deg \mathfrak{F} = \deg \mathfrak{F}^\diamond$. Set $\mathfrak{V}_2 := (df_1 \wedge \cdots \wedge df_l)^\diamond$. Thus, both $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are nonzero sections of $\wedge^{n-l} T(\mathfrak{q}^*)$. Note that $\deg \mathfrak{V}_2 = \sum_i (\deg f_i - 1)$.

For a vector field $\mathfrak{v}$, let $\iota_{\mathfrak{v}}$ denote the contraction of a section of $\wedge^j T(\mathfrak{q}^*)$ with respect to $\mathfrak{v}$. Then

$$\iota_{df_j} \mathfrak{V}_2 = (df_1 \wedge \cdots \wedge df_l \wedge df_j)^\diamond = 0 \quad \text{for all } j \in \{1, \dots, l\}. \tag{1.3}$$

Since each f_j is a central element of the Poisson algebra $\Bbbk[\mathfrak{q}^*]$, we have $\iota_{df_j} \pi = 0$. It follows that

$$\iota_{df_j} \mathfrak{V}_1 = 0 \quad \text{for all } j \in \{1, \dots, l\}. \tag{1.4}$$

For $\xi \in \mathfrak{q}^*$, let $V_\xi \subset T_\xi(\mathfrak{q}^*) \simeq \mathfrak{q}^*$ denote the annihilator of the $\Bbbk$-linear span of $\{(df_i)_\xi \mid i = 1, \dots, l\}$. Consider the open non-empty subset $S := \{\eta \in \mathfrak{q}^* \mid (df_1)_\eta \wedge \dots \wedge (df_l)_\eta \neq 0\}$. If $\xi \in S$, then $\dim V_\xi = n - l$. Let $t \in \wedge^{n-l} T_\xi(\mathfrak{q}^*)$ be an $(n-l)$-vector such that $\iota_{(df_i)_\xi} t = 0$, $i = 1, \dots, l$. Using the undergraduate linear algebra, one readily shows that $t \in \wedge^{n-l} V_\xi$. Applying this to Eqs. (1.3) and (1.4), we see that, for each $\xi \in S$, $(\mathfrak{V}_1)_\xi$ and $(\mathfrak{V}_2)_\xi$ belong to the same one-dimensional space $\wedge^{n-l} V_\xi \subset \wedge^{n-l}\mathfrak{q}^*$.

Thus, $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are two elements of a free $\Bbbk[\mathfrak{q}^*]$-module (the module of regular sections of $\wedge^{n-l} T(\mathfrak{q}^*)$), which is isomorphic to $\wedge^{n-l}\mathfrak{q}^* \otimes \Bbbk[\mathfrak{q}^*]$. Furthermore, $(\mathfrak{V}_1)_\xi$ and $(\mathfrak{V}_2)_\xi$ are linearly dependent as elements of $\wedge^{n-l}\mathfrak{q}^*$ for any $\xi \in S$. It then follows that $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are linearly dependent as elements of the vector space $\wedge^{n-l}\mathfrak{q}^* \otimes \Bbbk(\mathfrak{q}^*)$ over the field $\Bbbk(\mathfrak{q}^*)$. Hence there are mutually prime $F_1, F_2 \in \Bbbk[\mathfrak{q}^*]$ such that $F_1\mathfrak{V}_1 = F_2\mathfrak{V}_2$. If F_2 was non-constant, then the section $\mathfrak{V}_1$ would vanish on a divisor, which contradicts the codim-2 property. Therefore, we may assume that $F_2 \equiv 1$.

The equality $F_1\mathfrak{V}_1 = \mathfrak{V}_2$ shows that $\deg \mathfrak{V}_1 \leqslant \deg \mathfrak{V}_2$, that is, $(n-l)/2 \leqslant \sum_{i=1}^{l} \deg(f_i - 1)$, which yields (i).

If $\sum_{i=1}^{l} \deg f_i = (n+l)/2$, then $\deg F_1 = 0$, i.e., F_1 is a nonzero constant. Therefore $\mathfrak{q}^*_{\mathrm{reg}} = S$. Since $\operatorname{codim}(\mathfrak{q}^* \setminus S) \geqslant 2$, Theorem 1.5 below and the fact that $\operatorname{trdeg} \Bbbk[\mathfrak{q}^*]^Q = l$ guarantee us that $\Bbbk[\mathfrak{q}^*]^Q = \Bbbk[f_1, \dots, f_l]$. □

The following general result appears in [14, Theorem 1.1]. Its prototype is a theorem of Skryabin on algebras of invariants in a positive characteristic [19, Theorem 5.4].

1.5. Theorem. *Let V be a $\Bbbk$-vector space. Suppose that homogeneous polynomials $f_1, \dots, f_m \in \Bbbk[V]$ satisfy the property that* $\operatorname{codim}_V\{v \in V \mid (df_1)_v \wedge \dots \wedge (df_m)_v = 0\} \geqslant 2$. *Then any $f \in \Bbbk[V]$ that is algebraic over the subalgebra $\Bbbk[f_1, \dots, f_m]$ is necessarily contained in $\Bbbk[f_1, \dots, f_m]$.*

1.6. Remarks. (1) In the spirit of [10], Theorem 1.2 can be stated in the more general context of polynomial Poisson algebras and their centres.

(2) The equality

$$F_1 \cdot (\underbrace{\pi \wedge \pi \wedge \dots \wedge \pi}_{(n-l)/2}) = (df_1 \wedge \dots \wedge df_l)^\diamond$$

can be expressed in the coordinate form as follows. Let $x_1, \dots, x_n$ be a basis for $\mathfrak{q}$. Form the $n \times n$ matrix $\Pi = ([x_i, x_j])$ with entries in $\mathfrak{q} = \Bbbk[\mathfrak{q}^*]_1$. It is nothing but the matrix of the Poisson tensor π. If $I \subset [n]$ and $\#I = n - l$, then Π_I denotes the pfaffian of the principal $n - l$ submatrix of Π corresponding to I. That is, $\Pi_I = \operatorname{Pf}(([x_i, x_j])_{i,j \in I})$, and it is a polynomial of degree $(n-l)/2$. Another ingredient is the $l \times n$ matrix $\mathcal{D} = (\partial f_i / \partial x_j)$ of all partial derivatives of the polynomials $f_1, \dots, f_l$. Given I as above, set $\bar{I} = [n] \setminus I$. Let $\mathcal{D}_I$ denote the l minor of $\mathcal{D}$ whose set of columns is $\bar{I}$. Then we have

$$F_1 \Pi_I = \mathcal{D}_I \quad \text{for any } I \subset [n] \text{ with } \#I = n - l.$$

Similar (although more complicated) equalities for minors were obtained in [11, §1] for semi-simple Lie algebras.

(3) The proof of Theorem 1.2 shows that the equality

$$F_1 \cdot (\underbrace{\pi \wedge \pi \wedge \cdots \wedge \pi}_{(n-l)/2}) = F_2 \cdot (df_1 \wedge \cdots \wedge df_l)^{\diamond}$$

with some $F_1, F_2 \in \Bbbk[\mathfrak{q}^*]^Q$ holds for any Lie algebra. This allows to draw different conclusions under different assumptions. For instance, if $\mathfrak{q}$ is arbitrary and $f_1, \dots, f_l \in \Bbbk[\mathfrak{q}^*]^Q$ have the property that $l = \operatorname{ind}\mathfrak{q}$ and S is big, then $\sum_i \deg f_i \leqslant (\dim\mathfrak{q} + \operatorname{ind}\mathfrak{q})/2$.

Example. Let $\mathfrak{h} = \Bbbk a + \Bbbk b + \Bbbk h$ be a Heisenberg Lie algebra ($[a, b] = h$ and h is central). Here $\operatorname{ind}\mathfrak{h} = 1$ and $\Bbbk[\mathfrak{h}^*]^H$ is generated by $f_1 = h$. Hence $1 = \deg f_1 < (\dim\mathfrak{h} + \operatorname{ind}\mathfrak{h})/2 = 2$. This means that $\mathfrak{h}$ does not have the codim-2 property, which is also easily verified directly.

2. Semi-direct products, isotropy contractions, and $\mathbb{Z}_2$-gradings

Let Q be a connected algebraic group with Lie algebra $\mathfrak{q}$.

(A) *Semi-direct products*. Let V be a (finite-dimensional rational) Q-module, and hence a $\mathfrak{q}$-module. Then $\mathfrak{q} \times V$ has a natural structure of Lie algebra, V being an Abelian ideal in it. Explicitly, if $x, x' \in \mathfrak{q}$ and $v, v' \in V$, then

$$\big[(x, v), (x', v')\big] = \big([x, x'], x \cdot v' - x' \cdot v\big).$$

This Lie algebra is denoted by $\mathfrak{q} \ltimes V$. A connected algebraic group with Lie algebra $\mathfrak{q} \ltimes V$ is identified set-theoretically with $Q \times V$, and we write $Q \ltimes V$ for it. The product in $Q \ltimes V$ is given by

$$(s, v)(s', v') = \big(ss', (s')^{-1} \cdot v + v'\big).$$

In particular, $(s, v)^{-1} = (s^{-1}, -s \cdot v)$. The adjoint representation of $Q \ltimes V$ is given by the formula

$$\big(\mathrm{Ad}(s, v)\big)(x', v') = \big(\mathrm{Ad}(s)x', s \cdot v' - x' \cdot v\big), \tag{2.1}$$

where $v, v' \in V$, $x \in \mathfrak{q}$, and $s \in Q$.

Note that V can be regarded as either a commutative unipotent subgroup of $Q \ltimes V$ or a commutative nilpotent subalgebra of $\mathfrak{q} \ltimes V$. Referring to V as subgroup of $Q \ltimes V$, we write $1 \ltimes V$.

Set $\mathfrak{k} = \mathfrak{q} \ltimes V$ and $K = Q \ltimes V$. The dual space $\mathfrak{k}^*$ is identified with $\mathfrak{q}^* \oplus V^*$, and a typical element of it is denoted by $\eta = (\alpha, \xi)$. The coadjoint representation of $\mathfrak{k}$ is given by

$$\big(\mathrm{ad}^*(x, v)\big)(\alpha, \xi) = \big(\mathrm{ad}^*(x)\alpha - v * \xi, x \cdot \xi\big). \tag{2.2}$$

Here the mapping $((x, \xi) \in \mathfrak{q} \times V^*) \mapsto (x \cdot \xi \in V^*)$ is the natural $\mathfrak{q}$-module structure on V^*, and $((v, \xi) \in V \times V^*) \mapsto (v * \xi \in \mathfrak{q}^*)$ is the moment mapping with respect to the symplectic structure on $V \times V^*$.

2.3. Theorem. *Let $\mathfrak{k} = \mathfrak{q} \ltimes V$ be an arbitrary semi-direct product. Then*

(1) *The algebras* $\Bbbk[\mathfrak{k}]^K$ *and* $\Bbbk[\mathfrak{k}^*]^K$ *are bi-graded*;
(2) *There are natural inclusions* $i_{\mathfrak{q}}:\Bbbk[\mathfrak{q}]^Q \hookrightarrow \Bbbk[\mathfrak{k}]^K$ *and* $i_{V^*}:\Bbbk[V^*]^Q \hookrightarrow \Bbbk[\mathfrak{k}^*]^K$;
(3) *Let* $J_1 \subset \Bbbk[\mathfrak{k}]^K$ *be the ideal of all bi-homogeneous polynomials having a positive degree with respect to* V. *Then* $\Bbbk[\mathfrak{k}]^K = i_{\mathfrak{q}}(\Bbbk[\mathfrak{q}]^Q) \oplus J_1$;
(4) *Let* $J_2 \subset \Bbbk[\mathfrak{k}^*]^K$ *be the ideal of all bi-homogeneous polynomials having a positive degree with respect to* $\mathfrak{q}^*$. *Then* $\Bbbk[\mathfrak{k}^*]^K = i_{V^*}(\Bbbk[V^*]^Q) \oplus J_2$.

Proof. (1) Let $\Bbbk[\mathfrak{k}^*]_{(a,b)}$ denote the space of bi-homogeneous polynomials of degree a (respectively b) with respect to $\mathfrak{q}^*$ (respectively V^*). Clearly, each $\Bbbk[\mathfrak{k}^*]_{(a,b)}$ is Q-stable. Given $v \in V$, let $D_{v,\mathfrak{k}^*}$ denote the derivation of $\Bbbk[\mathfrak{k}^*]$ corresponding to $(0, v) \in \mathfrak{k}$. Then Eq. (2.2) shows that $D_{v,\mathfrak{k}^*}(\Bbbk[\mathfrak{k}^*]_{(a,b)}) \subset \Bbbk[\mathfrak{k}^*]_{(a-1,b+1)}$. Hence if $f \in \Bbbk[\mathfrak{k}^*]^K$ is a homogeneous polynomial, then all its bi-homogeneous components are Q-invariant and the bi-homogeneous component of highest degree with respect to V^* is also $1 \ltimes V$-invariant. Then we argue by induction.

The similar argument works for $(\mathfrak{k}, \mathrm{ad})$. Here $D_{v,\mathfrak{k}}(\Bbbk[\mathfrak{k}]_{(a,b)}) \subset \Bbbk[\mathfrak{k}]_{(a+1,b-1)}$ and one has to consider the bi-homogeneous component of $f \in \Bbbk[\mathfrak{k}]^K$ having the maximal degree with respect to $\mathfrak{q}$.

(2) We can regard $\mathfrak{q}$ and V^* as $Q \ltimes V$-modules with trivial action of $1 \ltimes V$. Then consider the natural surjective homomorphisms of $Q \ltimes V$-modules $\mathfrak{q} \ltimes V \to \mathfrak{q}$ and $(\mathfrak{q} \ltimes V)^* \to V^*$.

(3), (4) Obvious. □

We will omit the indication of $i_{\mathfrak{q}}$ and i_{V^*} in the sequel. If $\mathfrak{q} = \mathfrak{g}$ is a reductive (algebraic) Lie algebra, then *Inv*$(\mathfrak{g} \ltimes V, \mathrm{ad})$ is always polynomial [13, Theorem 6.2]. This is, however, not always the case for *Inv*$(\mathfrak{g} \ltimes V, \mathrm{ad}^*)$. For, it follows from Theorem 2.3(4) that any minimal generating system of $\Bbbk[V^*]^G$ is a part of a minimal generating system of *Inv*$(\mathfrak{g} \ltimes V, \mathrm{ad}^*)$. In particular, if *Inv*$(\mathfrak{g} \ltimes V, \mathrm{ad}^*)$ is polynomial, then so is $\Bbbk[V^*]^G$.

(B) *Isotropy contractions*. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{q}$ such that $\mathfrak{q} = \mathfrak{h} \oplus \mathfrak{m}$ for some ad $\mathfrak{h}$-stable subspace $\mathfrak{m} \subset \mathfrak{q}$. (Such an $\mathfrak{h}$ is said to be *reductive in* $\mathfrak{q}$.) Then $\mathfrak{m}$ is an H-module. If $\mathfrak{h}$ is the fixed-point subalgebra of an involutory automorphism of $\mathfrak{q}$, then it is reductive in $\mathfrak{q}$. In this case, $\mathfrak{h}$ is called a *symmetric subalgebra* of $\mathfrak{q}$ and the $(\mathfrak{q}, \mathfrak{h})$ is called a *symmetric pair*.

2.4. Definition. If $\mathfrak{h}$ is reductive in $\mathfrak{q}$, then the representation of H on $\mathfrak{m}$ is called the *isotropy representation* and the Lie algebra $\mathfrak{h} \ltimes \mathfrak{m}$ is called an *isotropy contraction* of $\mathfrak{q}$. If $\mathfrak{h}$ is symmetric, so the decomposition $\mathfrak{q} = \mathfrak{h} \oplus \mathfrak{m}$ is a $\mathbb{Z}_2$-grading, then $\mathfrak{h} \ltimes \mathfrak{m}$ is also called a $\mathbb{Z}_2$*-contraction* of $\mathfrak{q}$.

Here $\mathfrak{h} \ltimes \mathfrak{m}$ is a contraction of $\mathfrak{q}$ in the sense of the deformation theory of Lie algebras, see e.g. [24, Chapter 7, §2]. More precisely, consider the invertible linear map $c_t : \mathfrak{q} \to \mathfrak{q}$, $t \in \Bbbk \setminus \{0\}$, such that $c_t(h + m) = h + t^{-1}m$ ($h \in \mathfrak{h}$, $m \in \mathfrak{m}$). Define the new Lie algebra multiplication $[\,,\,]_{(t)}$ on the vector space $\mathfrak{q}$ by the rule

$$[x, y]_{(t)} := c_t\big(\big[c_t^{-1}(x), c_t^{-1}(y)\big]\big), \quad x, y \in \mathfrak{q}.$$

Then the algebras $\mathfrak{q}_{(t)}$ are isomorphic for all $t \neq 0$, and $\lim_{t\to 0} \mathfrak{q}_{(t)} = \mathfrak{h} \ltimes \mathfrak{m}$.

Suppose $\mathfrak{q} = \mathfrak{g}$ is reductive and $\mathfrak{h} \subset \mathfrak{g}$ is also reductive. Then the isotropy representation of H is orthogonalisable (in particular, $\mathfrak{m} \simeq \mathfrak{m}^*$ as H-module) and $\mathfrak{k} := \mathfrak{h} \ltimes \mathfrak{m}$ is called a *reductive isotropy contraction* (of $\mathfrak{g}$). Here $K^u := 1 \ltimes \mathfrak{m}$ is the unipotent radical of $K = H \ltimes \mathfrak{m}$.

A natural hope is that the algebras $\Bbbk[\mathfrak{k}]^K$ and $\Bbbk[\mathfrak{k}^*]^K$ could keep some good properties of $\Bbbk[\mathfrak{g}]^G$. But this is not always the case. For instance, the transcendence degree of $\Bbbk[\mathfrak{k}]^K$ and

$\Bbbk[\mathfrak{k}^*]^K$ can be larger than $\operatorname{rk}\mathfrak{g}$, and to guarantee equalities, one has to impose different constraints on $\mathfrak{h}$. Since $\mathfrak{h}$ is reductive, $\Bbbk[\mathfrak{k}]^K$ is polynomial [13, Theorem 6.2]; in other words, $\mathfrak{k}/\!\!/K$ is an affine space. For future reference, we record the following fact.

2.5. Proposition. *(See [13, Proposition 9.3].) Let $\mathfrak{k}$ be a reductive isotropy contraction of $\mathfrak{g}$. Then*

(1) $\dim \mathfrak{k}/\!\!/K = \operatorname{rk}\mathfrak{g}$ *if and only if $\mathfrak{h}$ contains a regular semisimple element of $\mathfrak{g}$;*
(2) $\operatorname{ind}\mathfrak{k} = \operatorname{rk}\mathfrak{g}$ *if and only if G/H is a spherical homogeneous space.*

Both these conditions are satisfied if $\mathfrak{h}$ is a symmetric subalgebra of $\mathfrak{g}$. However, the adjoint and coadjoint representations of $\mathfrak{k}$ are quite different, and should be studied separately.

2.6. Lemma. *If $\mathfrak{k} = \mathfrak{h} \ltimes \mathfrak{m}$ is a reductive isotropy contraction of $\mathfrak{g}$, then the quotient field of $\Bbbk[\mathfrak{k}^*]^K$ equals $\Bbbk(\mathfrak{k}^*)^K$.*

Proof. We have $K = T_H \cdot (K,K)$, where T_H is connected centre of H and (K,K) is the derived group of K. Since (K,K) has no rational character, the quotient field of $\Bbbk[\mathfrak{k}^*]^{(K,K)}$ equals $\Bbbk(\mathfrak{k}^*)^{(K,K)}$. It follows that any $f \in \Bbbk(\mathfrak{k}^*)^K$ can be written as $f = f_1/f_2$, where $f_1, f_2 \in \Bbbk[\mathfrak{k}^*]^{(K,K)}$ are semi-invariants of T_H of the same weight, say χ. Clearly, if $\Bbbk[\mathfrak{k}^*]$ contains a semi-invariant of T_H of weight ν, then it also contains a semi-invariant of weight $-\nu$. (Because $\mathfrak{k}^* \simeq \mathfrak{g}$ as T_H-modules.) The same assertion is also true for $\Bbbk[\mathfrak{k}^*]^{(K,K)}$ in place of $\Bbbk[\mathfrak{k}^*]$. [Use the fact that the automorphism of K (as a variety!) that is trivial on (K,K) and takes t to t^{-1} for any $t \in T_H$ does not change the K-action on $\mathfrak{k}^*$.] Thus, if $h \in \Bbbk[\mathfrak{k}^*]^{(K,K)}$ is a semi-invariant of weight $-\chi$, then $f = (f_1h)/(f_2h)$, and we are done. □

(C) $\mathbb{Z}_2$-*gradings of reductive Lie algebras*. Let G be a connected reductive algebraic group with $\mathfrak{g} = \operatorname{Lie} G$. Let $\mathcal{N}$ denote the set of nilpotent elements of $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-grading of $\mathfrak{g}$, then G_0 is the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. Recall some results on the isotropy representation $(G_0 : \mathfrak{g}_1)$. The standard reference for this is [8].

- Any $v \in \mathfrak{g}_1$ admits a unique decomposition $v = v_s + v_n$, where $v_s \in \mathfrak{g}_1$ is semisimple and $v_n \in \mathcal{N} \cap \mathfrak{g}_1$; $v = v_s$ if and only if $G_0 \cdot v$ is closed; $v = v_n$ if and only if the closure of $G_0 \cdot v$ contains the origin. For any $v \in \mathfrak{g}_1$, there is the induced $\mathbb{Z}_2$-grading of the centraliser $\mathfrak{g}_v = \mathfrak{g}_{0,v} \oplus \mathfrak{g}_{1,v}$, and $\dim \mathfrak{g}_0 - \dim \mathfrak{g}_{0,v} = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_{1,v}$.
- Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a maximal subspace consisting of pairwise commuting semisimple elements. Any such subspace is called a *Cartan subspace*. All Cartan subspaces are G_0-conjugate and $G_0 \cdot \mathfrak{c}$ is dense in $\mathfrak{g}_1$; $\dim \mathfrak{c}$ is called the *rank* of the $\mathbb{Z}_2$-grading or pair $(\mathfrak{g}, \mathfrak{g}_0)$, denoted $\operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0)$. If $v \in \mathfrak{c}$ is G_0-regular (i.e., $\dim(G_0 \cdot v)$ is maximal), then $\mathfrak{g}_{1,v} = \mathfrak{c}$ and $\mathfrak{g}_{0,v}$ is a generic stabiliser for the G_0-module $\mathfrak{g}_1$.
- The algebra $\Bbbk[\mathfrak{g}_1]^{G_0}$ is polynomial and $\dim \mathfrak{g}_1/\!\!/G_0 = \operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0)$. The quotient map $\pi : \mathfrak{g}_1 \to \mathfrak{g}_1/\!\!/G_0$ is equidimensional. We write $\mathfrak{N}(\mathfrak{g}_1)$ for $\pi^{-1}(\pi(0))$. Any fibre of π contains finitely many G_0-orbits and each closed G_0-orbit in $\mathfrak{g}_1$ meets $\mathfrak{c}$. There is a finite reflection group $W_\mathfrak{c} \subset GL(\mathfrak{c})$ ("the little Weyl group") such that $\mathfrak{c}/W_\mathfrak{c} \simeq \mathfrak{g}_1/\!\!/G_0$.

(D) *Reductive* $\mathbb{Z}_2$-*contractions*. Given a $\mathbb{Z}_2$-grading of $\mathfrak{g}$, consider the $\mathbb{Z}_2$-contraction $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$. Set $K^u := 1 \ltimes \mathfrak{g}_1$. The adjoint representation of $\mathfrak{k}$ was studied in [13]. Below we summarise the relevant invariant-theoretic results, see [13, Proposition 5.3 and Theorem 9.13]:

- Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ and $\mathfrak{g}_1^0$ the centraliser of $\mathfrak{t}_0$ in $\mathfrak{g}_1$. Then $\mathfrak{t}_0 \ltimes \mathfrak{g}_1^0$ is a generic stabiliser for $(\mathfrak{k}, \mathrm{ad})$. [As is well known, $\mathfrak{g}_0$ contains regular semisimple elements of $\mathfrak{g}$. Therefore $\dim(\mathfrak{t}_0 \ltimes \mathfrak{g}_1^0) = \mathrm{rk}\,\mathfrak{g}$.]
- $\Bbbk[\mathfrak{k}]^{K^u}$ is a polynomial algebra of Krull dimension $\dim \mathfrak{g}_0 + \dim \mathfrak{g}_1^0$.
- $\Bbbk[\mathfrak{k}]^K$ is a polynomial algebra of Krull dimension $\mathrm{rk}\,\mathfrak{g}$.
- the quotient map $\pi_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{k}/\!\!/K$ is equidimensional and $\Bbbk[\mathfrak{k}]$ is a free $\Bbbk[\mathfrak{k}]^K$-module.
- $\mathcal{N}(\mathfrak{k}) := \pi_{\mathfrak{k}}^{-1}(\pi_{\mathfrak{k}}(0))$ is an irreducible complete intersection. If $\Bbbk[\mathfrak{k}]^K = \Bbbk[f_1, \dots, f_l]$, $l = \mathrm{rk}\,\mathfrak{g}$, then the ideal of $\mathcal{N}(\mathfrak{k})$ in $\Bbbk[\mathfrak{k}]$ is generated by $f_1, \dots, f_l$.

However, the key fact is that there is a natural description of basic invariants in $\Bbbk[\mathfrak{k}]^K$ (see [13, Section 6]), which enables us to prove the above results. Namely, the set of basic invariants consists of two parts. First, we take a set of basic invariants in $\Bbbk[\mathfrak{g}_0]^{G_0}$, say $f_1, \dots, f_m$. Here $m = \mathrm{rk}\,\mathfrak{g}_0$. Next, we consider the set, $\mathrm{Mor}_{G_0}(\mathfrak{g}_0, \mathfrak{g}_1)$, of all G_0-equivariant polynomial morphisms $\tau : \mathfrak{g}_0 \to \mathfrak{g}_1$. By a result of Kostant [7], $\mathrm{Mor}_{G_0}(\mathfrak{g}_0, \mathfrak{g}_1)$ is a free $\Bbbk[\mathfrak{g}_0]^{G_0}$-module of rank $\dim \mathfrak{g}_1^0 = l - m$. Given $F \in \mathrm{Mor}_{G_0}(\mathfrak{g}_0, \mathfrak{g}_1)$, define the polynomial $\widehat{F} \in \Bbbk[\mathfrak{k}]$ by $\widehat{F}(x_0, x_1) = \langle F(x_0), x_1 \rangle$. Here $x_i \in \mathfrak{g}_i$ and $\langle\, , \rangle$ stands for a non-degenerate G_0-invariant symmetric bilinear form on $\mathfrak{g}_1$. It is easily seen that $\widehat{F} \in \Bbbk[\mathfrak{k}]^K$. If $F_1, \dots, F_{l-m}$ is a basis for $\mathrm{Mor}_{G_0}(\mathfrak{g}_0, \mathfrak{g}_1)$, then $f_1, \dots, f_m, \widehat{F}_1, \dots, \widehat{F}_{l-m}$ is a set of basic invariants in $\Bbbk[\mathfrak{k}]^K$.

Remark. It seems that the reason for success in case of $(\mathfrak{k}, \mathrm{ad})$ is that $\mathfrak{g}_0$ always contains regular semisimple elements of $\mathfrak{g}$. We will see in Section 5 that if $\mathfrak{g}_1$ contains a regular semisimple element, then $\Bbbk[\mathfrak{k}^*]^K$ is polynomial and, moreover, there is a similar description of basic invariants and similar properties hold.

3. Constructing invariants for reductive $\mathbb{Z}_2$-contractions

From now on, $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$ is a reductive $\mathbb{Z}_2$-contraction and $K = G_0 \ltimes \mathfrak{g}_1$. Our primary goal is to study invariant-theoretic properties of the coadjoint representation of $\mathfrak{k}$. However, we also mention results for $(\mathfrak{k}, \mathrm{ad})$, if they are parallel to those for $(\mathfrak{k}, \mathrm{ad}^*)$ and are not contained in [13].

We identify the G-modules $\mathfrak{g}$ and $\mathfrak{g}^*$, using a non-degenerate G-invariant symmetric bilinear form on $\mathfrak{g}$. Moreover, $\mathfrak{g}_i$ and $\mathfrak{g}_i^*$ (and hence $\mathfrak{k}$ and $\mathfrak{k}^*$) are identified as G_0-modules. This means, for instance, that we can speak about a Cartan subspace of $\mathfrak{g}_1^*$ and that any $f \in \Bbbk[\mathfrak{g}]$ can also be regarded as function on $\mathfrak{k}$ or $\mathfrak{k}^*$. Usually, it is clear from the context whether $\mathfrak{g}_i$ is regarded as a subspace of $\mathfrak{g}$ or $\mathfrak{k}$ or $\mathfrak{k}^*$. (This makes no difference as long as only G_0-module structure is involved.) However, if we wish to stress that $\mathfrak{g}_i$ is regarded as subspace of $\mathfrak{k}^*$, then we write $\mathfrak{g}_i^*$ for it.

There is a natural procedure of getting elements of $\Bbbk[\mathfrak{k}]^K$ and $\Bbbk[\mathfrak{k}^*]^K$ via "$\mathbb{Z}_2$-degenerations" of G-invariants on $\mathfrak{g}$. Let $\Bbbk[\mathfrak{g}]_{(a,b)}$ denote the space of bi-homogeneous polynomials of degree a with respect to $\mathfrak{g}_0$ and degree b with respect to $\mathfrak{g}_1$.

Given a homogeneous polynomial $f \in \Bbbk[\mathfrak{g}]$ of degree n, let us decompose f into the sum of bi-homogeneous components $f = \sum_{i=k}^m f_i$, where $f_i \in \Bbbk[\mathfrak{g}]_{(n-i,i)}$ and it is assumed that $f_k, f_m \neq 0$. Then we set $f^\bullet := f_m$ and $f_\bullet := f_k$.

3.1. Proposition. *Suppose that $f \in \Bbbk[\mathfrak{g}]^G$ is homogeneous. Then*

(i) *regarding f as function on $\mathfrak{k}$, we have $f_\bullet \in \Bbbk[\mathfrak{k}]^K$;*
(ii) *regarding f as function on $\mathfrak{k}^*$, we have $f^\bullet \in \Bbbk[\mathfrak{k}^*]^K$.*

Proof. Clearly, each f_i is G_0-invariant. The derivation of $\Bbbk[\mathfrak{g}]$ corresponding to $x \in \mathfrak{g}_1$ is denoted by $D_{x,\mathfrak{g}}$. The commutator relations for the $\mathbb{Z}_2$-grading show that for $f_i \in \Bbbk[\mathfrak{g}]_{(n-i,i)}$, we have

$$D_{x,\mathfrak{g}}(f_i) \in \Bbbk[\mathfrak{g}]_{(n-i-1,i+1)} \oplus \Bbbk[\mathfrak{g}]_{(n-i+1,i-1)}.$$

Accordingly, we write $D_{x,\mathfrak{g}} = D_x^{(+1)} + D_x^{(-1)}$, where $D_x^{(+1)} : \Bbbk[\mathfrak{g}]_{(n-i,i)} \to \Bbbk[\mathfrak{g}]_{(n-i-1,i+1)}$. It follows that $D_x^{(+1)}(f_m) = 0$ and $D_x^{(-1)}(f_k) = 0$, if $f \in \Bbbk[\mathfrak{g}]^G$. The key observation is that $D_x^{(+1)} = D_{x,\mathfrak{k}^*}$ and $D_x^{(-1)} = D_{x,\mathfrak{k}}$. □

Part (ii) appears in [2]. However, for some particular cases, this construction of invariants of the coadjoint representation is considered in [16]. The passages $f \mapsto f^\bullet$ and $f \mapsto f_\bullet$ will be referred to as $\mathbb{Z}_2$-*degenerations* of (homogeneous) invariants in $\Bbbk[\mathfrak{g}]^G$. In this way, one obtains bi-graded subalgebras

$$\mathrm{gr}_\bullet\big(\Bbbk[\mathfrak{g}]^G\big) := \big\{f_\bullet \mid f \in \Bbbk[\mathfrak{g}]^G\big\} \subset \Bbbk[\mathfrak{k}]^K \quad \text{and} \quad \mathrm{gr}^\bullet\big(\Bbbk[\mathfrak{g}]^G\big) := \big\{f^\bullet \mid f \in \Bbbk[\mathfrak{g}]^G\big\} \subset \Bbbk\big[\mathfrak{k}^*\big]^K.$$

However, both inclusions can be strict. For, the $\mathbb{Z}_2$-degeneration preserves the usual degree of polynomials, but it is possible in many cases to point out an element of $\Bbbk[\mathfrak{k}]^K$ or $\Bbbk[\mathfrak{k}^*]^K$ whose degree does not occur as degree of elements of $\Bbbk[\mathfrak{g}]^G$. For instance, if $\mathrm{rk}\,\mathfrak{g}_0 = \mathrm{rk}\,\mathfrak{g}$, then $\Bbbk[\mathfrak{k}]^K \simeq \Bbbk[\mathfrak{g}_0]^{G_0}$. Clearly, $\Bbbk[\mathfrak{g}_0]^{G_0}$ has “more” elements than $\Bbbk[\mathfrak{g}]^G$. Examples for $\Bbbk[\mathfrak{k}^*]^K$ are discussed in Remark 4.3.

3.2. Remark. As is explained in Section 2(D), invariants of $(\mathfrak{k}, \mathrm{ad})$ can be constructed using the $\Bbbk[\mathfrak{g}_0]^{G_0}$-module (*module of covariants*) $\mathrm{Mor}_{G_0}(\mathfrak{g}_0, \mathfrak{g}_1)$. One might suggest that there was a similar procedure for $(\mathfrak{k}, \mathrm{ad}^*)$, which makes use of the module of covariants $\mathrm{Mor}_{G_0}(\mathfrak{g}_1^*, \mathfrak{g}_0)$. However, this does not always work. For $F \in \mathrm{Mor}_{G_0}(\mathfrak{g}_1^*, \mathfrak{g}_0)$, we can define $\widehat{F} \in \Bbbk[\mathfrak{k}^*]$ by $\widehat{F}(\xi_0, \xi_1) = \langle F(\xi_1), \xi_0\rangle$, where $\xi_i \in \mathfrak{g}_i^*$ and $\langle\, ,\rangle$ is a G_0-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}_0$. Obviously, $\widehat{F}$ is G_0-invariant. But its invariance relative to $K^u = 1 \ltimes \mathfrak{g}_1$ reduces to the condition that

$$F(\xi) \in \mathfrak{g}_{0,\xi} \quad \text{for all } \xi \in \mathfrak{g}_1^*.$$

This condition and G_0-equivariance of F show that $F(\xi)$ belong to the centre of $\mathfrak{g}_{0,\xi}$. That is, such a nonzero covariant may only exist if a generic stabiliser for the G_0-module $\mathfrak{g}_1^*$ has a non-trivial centre.

3.3. Theorem. *Any reductive $\mathbb{Z}_2$-contraction has the codim-2 property for* ad^*.

Proof. (a) We explicitly describe certain big open subset of $\mathfrak{k}^*$ that is contained in $\mathfrak{k}^*_{\mathrm{reg}}$.

Let $\eta = (\alpha, \xi) \in \mathfrak{k}^*$ be an arbitrary point, where $\alpha \in \mathfrak{g}_0^*$ and $\xi \in \mathfrak{g}_1^*$. Write $\mathfrak{g}_{0,\xi}$ for the stabiliser of ξ in $\mathfrak{g}_0$. Then $\mathfrak{g}_1 * \xi = \mathrm{Ann}(\mathfrak{g}_{0,\xi}) \subset \mathfrak{g}_0^*$ and therefore $\mathfrak{g}_0^*/\mathfrak{g}_1 * \xi \simeq \mathfrak{g}_{0,\xi}^*$. Using the last isomorphism, we let $\bar\alpha$ denote the image of α in $\mathfrak{g}_{0,\xi}^*$. By [13, Proposition 5.5],

$$\dim \mathfrak{k}_\eta = \mathrm{codim}_{\mathfrak{g}_1^*}(G_0 \cdot \xi) + \dim(\mathfrak{g}_{0,\xi})_{\bar\alpha}, \tag{3.4}$$

where the last summand refers to the stabiliser of $\bar\alpha$ with respect to the coadjoint representation of $\mathfrak{g}_{0,\xi}$.

Let $\Omega \subset \mathfrak{g}_1^*$ be the open subset of G_0-regular points, i.e.,

$$\Omega = \{\xi \in \mathfrak{g}_1^* \mid \dim(G_0 \cdot \xi) = \dim \mathfrak{g}_1 - \mathrm{rk}(\mathfrak{g}, \mathfrak{g}_0)\}.$$

It is easily seen that Ω is a big open subset. It follows from Eq. (3.4) that in order to obtain a K-regular point in $\mathfrak{k}^*$, it suffices to take a G_0-regular point $\xi \in \mathfrak{g}_1^*$ and then, if the equality $\mathrm{ind}\, \mathfrak{g}_{0,\xi} = \mathrm{rk}\, \mathfrak{g} - \mathrm{rk}(\mathfrak{g}, \mathfrak{g}_0)$ holds, to take an α such that $\bar{\alpha} \in \mathfrak{g}_{0,\xi}^*$ is a $G_{0,\xi}$-regular point. Let us prove that the set of such points (α, ξ) contains a big open subset of $\mathfrak{k}^*$.

Let $\pi : \mathfrak{g}_1^* \to \mathfrak{g}_1^* /\!\!/ G_0$ be the quotient mapping. Consider the Luna stratification of $\mathfrak{g}_1^* /\!\!/ G_0$ [9, III.2]. (Recall that $\nu, \nu' \in \mathfrak{g}_1^* /\!\!/ G_0$ belong to the same stratum, if the closed G_0-orbits in $\pi^{-1}(\nu)$ and $\pi^{-1}(\nu')$ are isomorphic as G_0-varieties.) An exposition of Luna's theory can also be found in [20]. Let $(\mathfrak{g}_1^* /\!\!/ G_0)_i$ be the union of all strata of codimension i. For instance, $(\mathfrak{g}_1^* /\!\!/ G_0)_0$ is the unique open stratum. Set $\Omega_i = \pi^{-1}((\mathfrak{g}_1^* /\!\!/ G_0)_i) \cap \Omega$. Since π is equidimensional and each fibre of π meets Ω, $\mathrm{codim}_{\mathfrak{g}_1^*} \Omega_i = i$. In particular, $\Omega_0 \cup \Omega_1$ is a big open subset of $\mathfrak{g}_1^*$ and hence $(\Omega_0 \cup \Omega_1) \times \mathfrak{g}_0^*$ is a big open subset of $\mathfrak{k}^*$. Let us prove that $\mathfrak{k}^*_{\mathrm{reg}} \cap ((\Omega_0 \cup \Omega_1) \times \mathfrak{g}_0^*)$ is still big.

If $\xi \in \Omega_0$, then ξ is semisimple and $\mathfrak{g}_{0,\xi}$ is reductive. Since $(\mathfrak{g}_{0,\xi}, \mathrm{ad}^*)$ has codim-2 property, the set $\mathfrak{k}^*_{\mathrm{reg}} \cap (\Omega_0 \times \mathfrak{g}_0^*)$ is big in $\Omega_0 \times \mathfrak{g}_0^*$ (but not in $\mathfrak{k}^*$!). To obtain a big subset of $\mathfrak{k}^*$, we have to check that $\mathfrak{k}^*_{\mathrm{reg}} \cap (\Omega_1 \times \mathfrak{g}_0^*)$ is dense in $\Omega_1 \times \mathfrak{g}_0^*$. In view of the previous discussion, this amounts to the verification of the equality $\mathrm{ind}\, \mathfrak{g}_{0,\xi} = \mathrm{rk}\, \mathfrak{g} - \mathrm{rk}(\mathfrak{g}, \mathfrak{g}_0)$ for any $\xi \in \Omega_1$.

Using the Jordan decomposition in $\mathfrak{g}_1^*$ and taking the centraliser of the semisimple part of $\xi \in \Omega_1$, one reduces the problem to the case of symmetric pairs of rank 1. Namely, let $\xi = \xi_s + \xi_n$, where the semisimple element ξ_s belongs to a fixed Cartan subspace $\mathfrak{c}$. Then the centraliser of ξ_s in $\mathfrak{g}$ has the following structure: $\mathfrak{z}_{\mathfrak{g}}(\xi_s) = \mathfrak{a} \dotplus \mathfrak{h}$, where $\mathfrak{a} \subset \mathfrak{c}$, $\dim \mathfrak{a} = \dim \mathfrak{c} - 1$, $\mathfrak{h}$ is reductive, and the induced $\mathbb{Z}_2$-grading of $\mathfrak{h}$ has rank 1. Furthermore, $\xi_n \in \mathfrak{h}_1 \subset \mathfrak{h}$ and $\mathfrak{g}_{0,\xi} = \mathfrak{h}_{0,\xi_n}$. Hence it remains to handle the rank one case.

(b) Suppose $\mathrm{rk}(\mathfrak{g}, \mathfrak{g}_0) = 1$, i.e., $\dim \mathfrak{g}_1^* /\!\!/ G_0 = 1$. Then $(\mathfrak{g}_1^* /\!\!/ G_0)_1 = \{pt\} = \pi(0)$ and Ω_1 is the set of G_0-regular nilpotent elements of $\mathfrak{g}_1$. Here we have to check that if $\xi \in \Omega_1$, then $\mathrm{ind}\, \mathfrak{g}_{0,\xi} = \mathrm{rk}\, \mathfrak{g} - 1$. Since $\mathfrak{g}_\xi = \Bbbk\xi \dotplus \mathfrak{g}_{0,\xi}$ (a direct sum of Lie algebras), we need actually the equality $\mathrm{ind}\, \mathfrak{g}_\xi = \mathrm{rk}\, \mathfrak{g}$. Such an equality is known as "Elashvili's conjecture," and it is proved for all ξ in the classical Lie algebras in [25]. The only non-classical symmetric pair of rank one is $(\mathbf{F}_4, \mathbf{B}_4)$, where one has to test the stabiliser of a sole nilpotent G_0-orbit. Here the isotropy representation is the spinor representation of $\mathbf{B}_4$. By Igusa's computations [6], the stabiliser $\mathfrak{g}_{0,\xi}$ is the semi-direct product of $\mathbf{G}_2$ and its 7-dimensional representation. Then using Raïs' formula [15], we obtain $\mathrm{ind}\, \mathfrak{g}_{0,\xi} = 3$. (It is also easy to perform similar verifications for the three classical series of symmetric pairs of rank one.) □

Combining Proposition 2.5(2), Lemma 2.6, Theorems 1.2 and 3.3, we obtain

3.5. Corollary. *If $f_1, \ldots, f_l \in \Bbbk[\mathfrak{k}^*]^K$ are homogeneous algebraically independent and $l = \mathrm{rk}\, \mathfrak{g}$, then $\sum_{i=1}^{l} \deg f_i \geqslant (\dim \mathfrak{g} + l)/2$.*

As $\Bbbk[\mathfrak{k}^*]^K$ is a bi-graded algebra (Theorem 2.3), one can take bi-homogeneous polynomials $f_1, \ldots, f_l$. Our next goal is to provide a "bi-graded" refinement of Corollary 3.5.

For $f \in \Bbbk[\mathfrak{k}^*]_{(a,b)}$, we write $\mathrm{bideg}\, f = (a, b)$. Here a and b refer to the $\mathfrak{g}_0^*$-degree and $\mathfrak{g}_1^*$-degree, respectively. Let $\mathfrak{s} \subset \mathfrak{g}_0$ be a generic stabiliser for the isotropy representation $(G_0 : \mathfrak{g}_1)$. It is a reductive Lie algebra.

3.6. Theorem. *Suppose that* $f_1, \ldots, f_l \in \Bbbk[\mathfrak{k}^*]^K$ *are bi-homogeneous algebraically independent and* $l = \operatorname{rk}\mathfrak{g}$. *Then* $\sum_{i=1}^l \operatorname{bideg} f_i \geqslant ((\dim\mathfrak{s} + \operatorname{rk}\mathfrak{s})/2, \dim\mathfrak{g}_1)$ (*componentwise*).

Proof. First of all, the inequality in question is a refinement of that in Corollary 3.5. Indeed,

$$\dim\mathfrak{g}_1/\!\!/G_0 = \dim\mathfrak{g}_1 - \max_{x\in\mathfrak{g}_1}\dim(G_0\cdot x) = \dim\mathfrak{g}_1 - \dim\mathfrak{g}_0 + \dim\mathfrak{s}.$$

On the other hand, $\dim\mathfrak{g}_1/\!\!/G_0 = \dim\mathfrak{c} = \operatorname{rk}\mathfrak{g} - \operatorname{rk}\mathfrak{s}$. Equating two expressions for $\dim\mathfrak{g}_1/\!\!/G_0$ and rearranging them, we obtain $\dim\mathfrak{g}_1 + (\dim\mathfrak{s} + \operatorname{rk}\mathfrak{s})/2 = (\dim\mathfrak{g} + \operatorname{rk}\mathfrak{g})/2$, as required.

To prove the inequality, we use the construction of Theorem 1.2 in the coordinate form, as described in Remark 1.6(2). Let us match $\mathfrak{s}$ and $\mathfrak{c}$ such that $\mathfrak{s}$ is the stabiliser in $\mathfrak{g}_0$ of a generic element $x \in \mathfrak{c}$. Then $\mathfrak{s}\oplus\mathfrak{c} = \mathfrak{g}_x$ is a Levi subalgebra. Let $\mathfrak{t}(\mathfrak{s})$ be a Cartan subalgebra of $\mathfrak{s}$. By [13, Section 5], $\mathfrak{h} = \mathfrak{t}(\mathfrak{s}) \ltimes \mathfrak{c} \subset \mathfrak{k}$ is a generic stabiliser for $(\mathfrak{k}, \mathrm{ad}^*)$. We may (and will) consider $\mathfrak{h}$ as a subspace in either $\mathfrak{k}$ or $\mathfrak{k}^*$. In the last case we will denote it as $\mathfrak{h}^* = \mathfrak{t}(\mathfrak{s})^* \oplus \mathfrak{c}^*$. In our situation, $\mathfrak{h}$ has the property that $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{h}) = \mathfrak{h}$. It then follows from [13, Theorem 3.4] that $(\mathfrak{k}^*)^{\mathfrak{h}} = \mathfrak{h}^*$ and $\mathfrak{k} = [\mathfrak{k},\mathfrak{h}] \oplus \mathfrak{h}$. Taking the annihilators, we obtain the dual decomposition $\mathfrak{k}^* = \mathrm{ad}^*(\mathfrak{k})\cdot\xi \oplus \mathfrak{h}^*$, where $\xi \in \mathfrak{h}^*$ is a generic point.

Choose a basis $(x_1, \ldots, x_n)$ for $\mathfrak{k}$ such that $(x_1, \ldots, x_{n-l})$ is a basis for $[\mathfrak{k},\mathfrak{h}]$ and $(x_{n-l+1}, \ldots, x_n)$ is a basis for $\mathfrak{h}$. Recall that if $I \subset [n]$ and $\#I = n - l$, then $\Pi_I = \mathrm{Pf}(([x_i, x_j])_{i,j\in I})$ and $D_I = \det((\partial f_i/\partial x_j)_{j\notin I})$. Set $I_0 = [n-l]$ and consider Π_{I_0} and $\mathcal{D}_{I_0}$. More precisely, we need the restriction of these polynomials to the subspace $\mathfrak{h}^*$, $\bar\Pi_{I_0} = \Pi_{I_0}|_{\mathfrak{h}^*}$ and $\bar{\mathcal{D}}_{I_0} = \mathcal{D}_{I_0}|_{\mathfrak{h}^*}$. Clearly, $\bar{\mathcal{D}}_{I_0}$ is the Jacobian of $f_1|_{\mathfrak{h}^*}, \ldots, f_l|_{\mathfrak{h}^*}$. Hence $\operatorname{bideg}\bar{\mathcal{D}}_{I_0} = (\sum_{i=1}^l \operatorname{bideg} f_i) - (\operatorname{rk}\mathfrak{s}, \dim\mathfrak{c})$.

Claim 1. $\bar\Pi_{I_0} \neq 0$.

Proof. It is easily seen that $([x_i, x_j]|_{\mathfrak{h}^*})_{i,j\in I}$ has a zero column unless $I = I_0$, hence $\Pi_I|_{\mathfrak{h}^*} = 0$ unless $I = I_0$. The definition of generic stabilisers says that $K\cdot\mathfrak{h}^*$ is dense in $\mathfrak{k}^*$. Since $\pi^{(n-l)/2}$ is K-invariant and the functions Π_I, $I \subset [n]$, are the coefficients of $\pi^{(n-l)/2}$ in the basis $\{\bigwedge_{i\in I} x_i \mid I \subset [n]\}$, they all cannot vanish on $\mathfrak{h}^*$. □

Claim 2. $\operatorname{bideg}\bar\Pi_{I_0} = ((\dim\mathfrak{s} - \operatorname{rk}\mathfrak{s})/2, \dim\mathfrak{g}_1 - \dim\mathfrak{c})$.

Proof. Since $[\mathfrak{k},\mathfrak{h}] = (\mathfrak{h}^*)^\perp$, this space is a sum of its intersections with $\mathfrak{g}_0^*$ and $\mathfrak{g}_1^*$. More precisely, using non-degenerate G_0-invariant symmetric bilinear forms on $\mathfrak{g}_0$ and $\mathfrak{g}_1$, we obtain $[\mathfrak{k},\mathfrak{h}]_0 = \mathfrak{t}(\mathfrak{s})^\perp$ and $[\mathfrak{k},\mathfrak{h}]_1 = \mathfrak{c}^\perp$. Hence $\dim[\mathfrak{k},\mathfrak{h}]_0 = \dim\mathfrak{g}_0 - \operatorname{rk}\mathfrak{s}$ and $\dim[\mathfrak{k},\mathfrak{h}]_1 = \dim\mathfrak{g}_1 - \dim\mathfrak{c}$. Since $\dim\mathfrak{g}_0 - \dim\mathfrak{s} = \dim\mathfrak{g}_1 - \dim\mathfrak{c}$, we have $\dim[\mathfrak{k},\mathfrak{h}]_0 = \dim[\mathfrak{k},\mathfrak{h}]_1 + (\dim\mathfrak{s} - \operatorname{rk}\mathfrak{s})$. Assume that a basis for $[\mathfrak{k},\mathfrak{h}]$ is chosen such that we first have a basis for $\mathfrak{t}(\mathfrak{s})^\perp \cap \mathfrak{s}$, then a basis for $\mathfrak{s}^\perp \cap [\mathfrak{k},\mathfrak{h}]_0$, and finally a basis for $[\mathfrak{k},\mathfrak{h}]_1$. It is easily seen that, for this choice of a basis, the matrix $([x_i, x_j]|_{\mathfrak{h}^*})_{i,j\in I_0}$ is of the form:

$$\begin{pmatrix} A & 0 & 0 \\ 0 & * & B \\ 0 & -B^t & 0 \end{pmatrix},$$

where A is a skew-symmetric matrix of order $\dim\mathfrak{s} - \operatorname{rk}\mathfrak{s}$, with entries in $\mathfrak{g}_0$; B is a square matrix of order $\dim\mathfrak{g}_1 - \dim\mathfrak{c}$, with entries in $\mathfrak{g}_1$. It follows that $\bar\Pi_{I_0} = \mathrm{Pf}(A)\det(B)$ has the required bi-degree. □

By Remark 1.6(2), there is an $F \in \Bbbk[\mathfrak{k}^*]$ such that $F\Pi_I = \mathcal{D}_I$ for any $I \subset [n]$. Applying this to I_0 shows that bideg $\bar{\Pi}_{I_0} \leqslant$ bideg $\bar{\mathcal{D}}_{I_0}$, which completes the proof of theorem. □

4. Good generating systems for invariants associated with symmetric pairs

The presence of codim-2 property for the $\mathbb{Z}_2$-contractions and the procedure of $\mathbb{Z}_2$-degeneration of invariants enable us to state a helpful sufficient condition for the polynomiality of $\Bbbk[\mathfrak{k}^*]^K$.

4.1. Definition. Let $f_1, \dots, f_l \in \Bbbk[\mathfrak{g}]^G$ be a set of basic invariants. We say that it is a *good generating system* for $(\mathfrak{g}, \mathfrak{g}_0)$ if the $\mathbb{Z}_2$-degenerations $f_1^\bullet, \dots, f_l^\bullet \in \Bbbk[\mathfrak{k}^*]^K$ are algebraically independent.

4.2. Theorem. *If* $f_1, \dots, f_l$ *is a good generating system for* $(\mathfrak{g}, \mathfrak{g}_0)$, *then* (i) $\Bbbk[\mathfrak{k}^*]^K$ *is freely generated by* $f_1^\bullet, \dots, f_l^\bullet$ *and* (ii) $(df_1^\bullet)_\xi, \dots, (df_l^\bullet)_\xi$ *are linearly independent if and only if* $\xi \in \mathfrak{k}^*_{\mathrm{reg}}$. *Furthermore, in this case,* $\Bbbk[\mathfrak{k}^*]^K = \mathrm{gr}^\bullet(\Bbbk[\mathfrak{g}]^G)$.

Proof. Since $\deg f_i = \deg f_i^\bullet$, $\mathrm{rk}\,\mathfrak{g} = \mathrm{ind}\,\mathfrak{k}$, and $(\mathfrak{k}, \mathrm{ad}^*)$ has codim-2 property, Theorem 1.2(ii) applies to $f_1^\bullet, \dots, f_l^\bullet$. □

The property of being ‘good’ for a generating system is rather specific and can easily be disturbed. For instance, if $\mathfrak{g} = \mathfrak{so}_{2n+1}$, then the coefficients of the characteristic polynomial of a matrix $M \in \mathfrak{so}_{2n+1}$ form a good generating system for any symmetric pair $(\mathfrak{so}_{2n+1}, \mathfrak{so}_m \dotplus \mathfrak{so}_{2n+1-m})$ (see Theorem 4.4 below). But the polynomials $\mathrm{tr}(M^{2i})$, $i = 1, \dots, n$, do not form a good a generating system.

4.3. Remark. Good generating systems do not always exist. For instance, consider the symmetric pair $(\mathbf{E}_6, \mathbf{F}_4)$. Here $\Bbbk[\mathfrak{g}_1]^{G_0}$ is freely generated by two polynomials of degree 2 and 3. Since $\Bbbk[\mathfrak{g}_1]^{G_0} \hookrightarrow \Bbbk[\mathfrak{k}^*]^K$, the latter has an element of degree three. However, the basic degrees of $\mathbf{E}_6$ are $2, 5, 6, 8, 9, 12$. Hence $\Bbbk[\mathfrak{g}]^G$ does not contain elements of degree 3 and the equality $\Bbbk[\mathfrak{k}^*]^K = \mathrm{gr}^\bullet(\Bbbk[\mathfrak{g}]^G)$ cannot hold. Similar phenomenon occurs for three other symmetric pairs: $(\mathbf{E}_6, \mathbf{D}_5 \dotplus \mathfrak{t}_1)$, $(\mathbf{E}_7, \mathbf{E}_6 \dotplus \mathfrak{t}_1)$, $(\mathbf{E}_8, \mathbf{E}_7 \dotplus \mathbf{A}_1)$. These are precisely the symmetric pairs such that the restriction homomorphism $\Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{g}_1]^{G_0}$ is not onto [5].

For some symmetric pairs, it is possible to check directly that certain generating system is good. Below, we consider several examples.

Practical tricks. (1) To prove that some polynomials in $\Bbbk[\mathfrak{k}^*]^K$ are algebraically independent, it suffices to verify this for their restriction to a subspace. In case of $\mathbb{Z}_2$-contractions, it is convenient to take the subspace $\mathfrak{c} \oplus \mathfrak{s}$, where $\mathfrak{c}$ is a fixed Cartan subspace of $\mathfrak{g}_1$ and $\mathfrak{s} = \mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{c})$. Recall that $\mathfrak{s}$ is a generic stabiliser for the G_0-module $\mathfrak{g}_1$ and $\mathfrak{s} \oplus \mathfrak{c} = \mathfrak{g}_x$ for a generic $x \in \mathfrak{c}$. Following our convention, we also regard $\mathfrak{s} \oplus \mathfrak{c}$ as a subspace of $\mathfrak{k}^*$. Furthermore, one can work with the smaller subspace $\mathfrak{c} \oplus \mathfrak{t}(\mathfrak{s})$. Notice that this vector space has three masks: as subspace of $\mathfrak{g}$ it is just a Cartan subalgebra; as a subspace of $\mathfrak{k}$ it is a generic stabiliser for $(\mathfrak{k}, \mathrm{ad}^*)$, say $\mathfrak{h}$; as a subspace of $\mathfrak{k}^*$ it is the fixed point space of $\mathfrak{h}$.

(2) Another useful observation is that if $f \in \Bbbk[\mathfrak{g}]^G$, then taking the restriction of $f^\bullet$ to $\mathfrak{c} \oplus \mathfrak{s}$ (or $\mathfrak{c} \oplus \mathfrak{t}(\mathfrak{s})$) is the same as first restricting f to $\mathfrak{c} \oplus \mathfrak{s}$ (or $\mathfrak{c} \oplus \mathfrak{t}(\mathfrak{s})$) and then taking the component

of highest degree with respect to $\mathfrak{c}$. The reason is that $f^\bullet|_{\mathfrak{c}\oplus\mathfrak{t}(\mathfrak{s})} \neq 0$, since $f^\bullet \in \Bbbk[\mathfrak{k}^*]^K$ and $\mathrm{Ad}^* K \cdot (\mathfrak{c} \oplus \mathfrak{t}(\mathfrak{s}))$ is dense in $\mathfrak{k}^*$.

4.4. Theorem. *There is a good generating system for* $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{so}_{n+m}, \mathfrak{so}_m \dotplus \mathfrak{so}_n)$.

Proof. Here $l = \lfloor (n+m)/2 \rfloor$ and $\mathrm{rk}(\mathfrak{g}, \mathfrak{g}_0) = \min\{n, m\}$. We use the natural matrix model for $(\mathfrak{g}, \mathfrak{g}_0)$:

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix} \right\},$$

where A (respectively B) is a skew-symmetric matrix of order m (respectively n) and C is an $m \times n$ matrix. Assume that $m \leqslant n$. Then $\mathfrak{s} \simeq \mathfrak{so}_{n-m}$ and $\mathfrak{s} \subset \mathfrak{so}_n$.

For a Cartan subspace $\mathfrak{c}$, we take the set of matrices C with only nonzero entries along the diagonal starting in the upper left corner of C. Then $\mathfrak{s}$ is the lower-right submatrix of B of order $n - m$. That is, taking the partition of M into the nine submatrices corresponding to the sizes $m, m, n - m$, we obtain

$$\mathfrak{c} \oplus \mathfrak{s} = \left\{ \tilde{M} = \begin{pmatrix} 0 & D & 0 \\ -D & 0 & 0 \\ 0 & 0 & E \end{pmatrix} \right\},$$

where D is a diagonal matrix of order m and E is a skew-symmetric matrix of order $n - m$. Let $d_1, \ldots, d_m$ be the diagonal entries of D.

For a skew-symmetric matrix M, let $f_i(M)$ denote the sum of all principal $2i$-minors of M. If $n + m$ is odd (respectively even), then we take the basic invariants $f_1, \ldots, f_l$ (respectively $f_1, \ldots, f_{l-1}$, and the pfaffian Pf). Let us prove that they form a good generating system in $\mathit{Inv}(\mathfrak{so}_{n+m}, \mathrm{ad})$ for $(\mathfrak{g}, \mathfrak{g}_0)$.

It easily follows from the block structure of $\tilde{M}$ that $f_i|_{\mathfrak{c}\oplus\mathfrak{s}}$ has a monomial entirely in d_i's if and only if $i \leqslant m$. Furthermore, if $i > m$, then one can always find a monomial in $f_i|_{\mathfrak{c}\oplus\mathfrak{s}}$ whose degree with respect to d_i's equals $2m$. Thus,

$$\operatorname{bideg} f_i^\bullet = \begin{cases} (0, 2i) & \text{if } i \leqslant m, \\ (2i - 2m, 2m) & \text{if } i > m. \end{cases}$$

Likewise, for $n + m$ even, we have the pfaffian, and $\operatorname{bideg} \mathrm{Pf}^\bullet = ((n - m)/2, m)$. Actually, it is easily seen that

$$f_i^\bullet(\tilde{M}) = \begin{cases} \text{the } i\text{th elementary symmetric function in } d_1^2, \ldots, d_m^2, & i \leqslant m, \\ \left(\prod_{i=1}^{m} d_i^2\right) \cdot f_{i-m}(E), & i > m, \end{cases}$$

and $\mathrm{Pf}^\bullet(\tilde{M}) = (\prod_{i=1}^m d_i) \cdot \mathrm{Pf}(E)$. Consequently, the $f_i^\bullet|_{\mathfrak{c}\oplus\mathfrak{s}}$, $i = 1, \ldots, \lfloor (n+m-1)/2 \rfloor$ (together with $\mathrm{Pf}^\bullet|_{\mathfrak{c}\oplus\mathfrak{s}}$ if $n + m$ is even) are algebraically independent. □

The following case is rather similar, although a bit more involved.

4.5. Theorem. *There is a good generating system for* $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{gl}_{n+m}, \mathfrak{gl}_m \dotplus \mathfrak{gl}_n)$.

Proof. Here $l = n + m$ and $\operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0) = \min\{n, m\}$. We use the natural matrix model for $(\mathfrak{g}, \mathfrak{g}_0)$:

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & M_2 \\ M_3 & 0 \end{pmatrix} \right\},$$

where M_1 (respectively M_4) is a matrix of order m (respectively n), M_2 is an $m \times n$ matrix, and M_3 is an $n \times m$ matrix. Assume below that $n \geqslant m$. Then $\mathfrak{s} \simeq \mathfrak{gl}_{n-m} \dotplus \mathfrak{t}_m$.

Let us describe our choice of $\mathfrak{c} \subset \mathfrak{g}_1$ and thereby of $\mathfrak{s} = \mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{s})$. We take M_2, M_3 such that $M_2 = (B \quad 0)$ and $M_3 = \left(\begin{smallmatrix} -B \\ 0 \end{smallmatrix}\right)$, where B is an arbitrary diagonal $m \times m$ matrix.

Then taking the partition of M into the nine submatrices corresponding to the sizes $m, m, n - m$, we obtain

$$\mathfrak{c} \oplus \mathfrak{s} = \left\{ \tilde{M} = \begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & E \end{pmatrix} \right\}, \tag{4.6}$$

where A and B are diagonal matrices of order m and E is an arbitrary matrix of order $n - m$. Let $a_1, \ldots, a_m$ (respectively $b_1, \ldots, b_m$) be the diagonal entries of A (respectively B).

Let $f_i(M)$ denote the sum of all principal minors of order i of a square matrix M. Let us prove that $f_1, \ldots, f_{n+m}$ form a good generating system in $\mathit{Inv}(\mathfrak{gl}_{n+m}, \mathrm{ad})$. It easily follows from Eq. (4.6) that the restriction of f_i to $\mathfrak{c} \oplus \mathfrak{s}$ has a monomial entirely in b_i's if and only if i is even and $i \leqslant 2m$. If i is odd and $i < 2m$, then one can only find a monomial whose all but one indeterminates are some b_i's. One other indeterminate is either an a_j (where j depends on the b_i's chosen) or an arbitrary diagonal entry of E. Finally, if $i > 2m$, then one can always produce a monomial of $f_i|_{\mathfrak{c} \oplus \mathfrak{s}}$ whose degree with respect to b_i's equals $2m$. Thus,

$$\operatorname{bideg} f_i^\bullet = \begin{cases} (0, i) & \text{if } i \leqslant 2m \text{ and } i \text{ is even}, \\ (1, i-1) & \text{if } i < 2m \text{ and } i \text{ is odd}, \\ (i - 2m, 2m) & \text{if } i > 2m. \end{cases}$$

To describe these polynomials explicitly, we need some notation. Let σ_i denote the ith elementary symmetric function. Set

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}.$$

Then looking at the principal minors of $\tilde{M}$ and their highest components with respect to $\mathfrak{c}$ (i.e., B), one easily obtains

$$\begin{cases} f_{2i}^\bullet(\tilde{M}) = \sigma_i(b_1^2, \ldots, b_m^2), & i \leqslant m, \\ f_{2i+1}^\bullet(\tilde{M}) = \operatorname{tr}(E)\sigma_i(b_1^2, \ldots, b_m^2) + \operatorname{tr}(\mathcal{B}^{2i}\mathcal{A}), & i < m, \\ f_j^\bullet(\tilde{M}) = f_{j-2m}(E)\sigma_m(b_1^2, \ldots, b_m^2), & j > 2m. \end{cases}$$

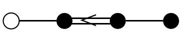
These formulae show that the polynomials $\{f_i^\bullet\}$ are algebraically independent. $\square$

Another example concerns an exceptional symmetric pair. This was obtained in collaboration with O. Yakimova.

4.7. Theorem. *For* $(\mathfrak{g}, \mathfrak{g}_0) = (\mathbf{F}_4, \mathbf{B}_4)$*, there is a good generating system. The bi-degrees of the basic invariants in* $\Bbbk[\mathfrak{k}^*]^K$ *are* $(0, 2)$, $(2, 4)$, $(4, 4)$, $(6, 6)$.

Sketch of the proof. We explicitly construct a good generating system, using ad hoc arguments. Here $\dim \mathfrak{c} = 1$, $\mathfrak{s} \simeq \mathbf{B}_3$, and we work with the restrictions of G-invariant functions to $\mathfrak{t} = \mathfrak{c} \oplus \mathfrak{t}(\mathfrak{s})$. The latter is a Cartan subalgebra of $\mathfrak{g}$ and, by virtue of Chevalley's restriction theorem, we actually deal with the Weyl group invariants on it. The Weyl group of $\mathbf{F}_4$, $W(\mathbf{F}_4)$, is a semi-direct product of the normal subgroup $W(\mathbf{D}_4)$ and $\mathcal{S}_3 = W(\mathbf{A}_2)$. Here $W(\mathbf{D}_4)$ is generated by the reflection with respect to the long roots of $\mathbf{F}_4$ and $W(\mathbf{A}_2)$ is generated by the reflections corresponding to the short simple roots of $\mathbf{F}_4$. Hence, to obtain $W(\mathbf{F}_4)$-invariants, one can take the invariants of $W(\mathbf{D}_4)$ and then consider the $\mathcal{S}_3$-action on them. We begin with a natural set of basic invariants of $W(\mathbf{D}_4)$. The $\mathcal{S}_3$-action has a rather bulky expression with respect to this set, but it is still a manageable task to write explicitly down the expressions for $W(\mathbf{F}_4)$-invariants through the $W(\mathbf{D}_4)$-invariants. Then, playing around with these invariants, we "correct" them on order to obtain a good generating system.

Here are the relevant data. We use the expressions for the simple roots of $\mathbf{F}_4$ and their numbering from [22]; that is, $\alpha_1 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$, $\alpha_2 = \varepsilon_4$, $\alpha_3 = \varepsilon_3 - \varepsilon_4$, and $\alpha_4 = \varepsilon_2 - \varepsilon_3$. Then $\Delta(\mathbf{D}_4) = \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j\}$. The Satake diagram of our symmetric pair is:



(The white node represents α_1.) This shows that the simple roots of $\mathfrak{s}$ are $\alpha_2, \alpha_3, \alpha_4$ and allows us to determine the splitting of $\mathfrak{t}_{\mathbb{R}}$. Here $\mathfrak{c}_{\mathbb{R}} = \mathbb{R}\varepsilon_1$ and $\mathfrak{t}(\mathfrak{s})_{\mathbb{R}} = \mathbb{R}\varepsilon_2 \oplus \mathbb{R}\varepsilon_3 \oplus \mathbb{R}\varepsilon_4$. The basic invariants of $W(\mathbf{D}_4)$ are:

$$
\begin{aligned}
f_2 &= \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2, \\
f_4' &= \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4, \\
f_4 &= \sum_{i<j} \varepsilon_i^2\varepsilon_j^2, \\
f_6 &= \sum_{i<j<k} \varepsilon_i^2\varepsilon_j^2\varepsilon_k^2.
\end{aligned}
$$

The group $\mathcal{S}_3$ is realised as the group generated by the reflections s_{α_1} and s_{α_2}. Using this, it straightforward to write down the $\mathcal{S}_3$-action on $W(\mathbf{D}_4)$-invariants and to determine the $W(\mathbf{F}_4)$-invariants. The key observation is that f_2 and $f_6 - \frac{1}{6}f_2 f_4$ are already $W(\mathbf{F}_4)$-invariants, and the plane $\mathrm{Span}\{f_4', 4f_4 - f_2^2\}$ affords the standard reflection representation of $\mathcal{S}_3$. The basic invariants of $\mathbf{F}_4$ have degrees $2, 6, 8, 12$. Here are the expressions of a good generating system g_2, g_6, g_8, g_{12} via the f_i's:

$$
\begin{aligned}
g_2 &= f_2, \\
g_6 &= f_6 - \frac{1}{6} f_2 f_4,
\end{aligned}
$$

$$g_8 = f_4'^2 + \frac{1}{12} f_4^2 - \frac{1}{4} f_2 f_6,$$

$$g_{12} = 4 f_4'^2 f_4 - \frac{3}{2} f_6^2 - \frac{3}{2} f_4'^2 f_2^2 - \frac{1}{9} f_4^3 + \frac{1}{2} f_2 f_4 f_6.$$

The highest components of these polynomials with the respect to $\mathfrak{c}$, i.e., with respect to ε_1 are:

$$g_2^\bullet = \varepsilon_1^2,$$

$$g_6^\bullet = \varepsilon_1^4 \left(\varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right),$$

$$g_8^\bullet = \varepsilon_1^4 \left(\frac{1}{12} \left(\varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)^2 - \frac{1}{4} \left(\varepsilon_2^2 \varepsilon_3^2 + \varepsilon_2^2 \varepsilon_4^2 + \varepsilon_3^2 \varepsilon_4^2\right) \right),$$

$$g_{12}^\bullet = \varepsilon_1^6 \left(-\frac{3}{2} \varepsilon_2^2 \varepsilon_3^2 \varepsilon_4^2 - \frac{1}{9} \left(\varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)^3 + \frac{1}{2} \left(\varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right) \left(\varepsilon_2^2 \varepsilon_3^2 + \varepsilon_2^2 \varepsilon_4^2 + \varepsilon_3^2 \varepsilon_4^2\right) \right).$$

It follows that these highest components are algebraically independent. □

It is likely that, for all symmetric pairs not mentioned in Remark 4.3, there is a good generating system. However, this is not easy to prove, even for the other classical series.

5. $\mathcal{N}$-regular $\mathbb{Z}_2$-gradings and their contractions

A $\mathbb{Z}_2$-grading (a symmetric pair) is said to be $\mathcal{N}$-*regular* if $\mathfrak{g}_1$ contains a regular nilpotent element of $\mathfrak{g}$. By [1], a $\mathbb{Z}_2$-grading is $\mathcal{N}$-regular if and only if $\mathfrak{g}_1$ contains a regular semisimple element if and only if any nilpotent G-orbit in $\mathfrak{g}$ meets $\mathfrak{g}_1$. (This is no longer true for $\mathbb{Z}_m$-gradings with $m > 2$.)

Until the end of this section, we assume that our $\mathbb{Z}_2$-grading is $\mathcal{N}$-regular. Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a Cartan subspace.

Set $Z_1 = \overline{G \cdot \mathfrak{g}_1} = \overline{G \cdot \mathfrak{c}}$. By [12, Theorem 4.7], Z_1 is a normal complete intersection in $\mathfrak{g}$ and the ideal of Z_1 in $\Bbbk[\mathfrak{g}]$ is generated by certain basic invariants. That is, there is a set of basic invariants $f_1, \ldots, f_l$ such that $f_i|_{Z_1} \equiv 0$ for $i \geqslant k+1$ and $\Bbbk[Z_1]^G$ is freely generated by $f_i|_{Z_1}$ for $i \leqslant k$. Furthermore, since the restriction map $\Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{g}_1]^{G_0}$ is onto [12, Theorem 3.5], $\Bbbk[\mathfrak{g}_1]^{G_0}$ is freely generated by $\bar{f_i} = f_i|_{\mathfrak{g}_1}$, $i \leqslant k$, and $k = \mathrm{rk}(\mathfrak{g}, \mathfrak{g}_0)$. Thus, each f_i, $i = 1, \ldots, k$, has the bi-homogeneous component that does not depend on $\mathfrak{g}_0$, whereas f_j, $j = k+1, \ldots, l$, does not have such a bi-homogeneous component.

If $x \in \mathfrak{g}_1 \cap \mathfrak{g}_{\mathrm{reg}}$, then $(df_i)_x$, $i = 1, \ldots, l$, are linearly independent [7]. Because f_j $(k+1 \leqslant j \leqslant l)$ does not have a component of degree 0 with respect to $\mathfrak{g}_0$, it must have a component of degree 1 with respect to $\mathfrak{g}_0$. Otherwise we would have $(df_j)_v = 0$ for any $v \in \mathfrak{g}_1$. The linear component of f_j with respect to $\mathfrak{g}_0$ can be written as $(x_0, x_1) \mapsto \langle x_0, F_j(x_1) \rangle$, where $0 \neq F_j \in \mathrm{Mor}(\mathfrak{g}_1, \mathfrak{g}_0)$ and $\deg F_j = \deg f_j - 1$. Since each bi-homogeneous component of f_j is G_0-invariant, F_j must be G_0-equivariant, i.e., $F_j \in \mathrm{Mor}_{G_0}(\mathfrak{g}_1, \mathfrak{g}_0)$, $j = k+1, \ldots, l$.

As $\dim \mathfrak{g}_1 /\!\!/ G_0 = k$, the $\mathcal{N}$-regularity implies that $\dim \mathfrak{g}_{1,x} = k$ and $\dim \mathfrak{g}_{0,x} = l-k$ whenever $x \in \mathfrak{g}_1 \cap \mathfrak{g}_{\mathrm{reg}}$. This also shows that $\dim \mathfrak{g}_1 - k = \dim \mathfrak{g}_0 - (l-k)$. In view of G_0-equivariance, $F_j(x) \in \mathfrak{g}_{0,x}$, and the linear independence of the differentials $(df_i)_x$ imply that $\{F_j(x)\}$ are linearly independent. Hence $\{F_j(x) \mid j = k+1, \dots, l\}$ is a basis for $\mathfrak{g}_{0,x}$ for any $x \in \mathfrak{g}_1 \cap \mathfrak{g}_{\mathrm{reg}}$. Thus, we obtain the following presentation of the basic invariants $f_1, \dots, f_l$:

$$\begin{cases} f_i(x_0, x_1) = \bar{f}_i(x_1) + (\text{terms of higher degree w.r.t. } x_0), & i \leqslant k, \\ f_j(x_0, x_1) = \langle x_0, F_j(x_1)\rangle + (\text{terms of higher degree w.r.t. } x_0), & j \geqslant k+1. \end{cases} \tag{5.1}$$

Set $\widehat{F}_j(x_0, x_1) = \langle x_0, F_j(x_1)\rangle$.

5.2. Theorem. *Let $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$ be the $\mathbb{Z}_2$-contraction of an $\mathcal{N}$-regular $\mathbb{Z}_2$-grading of rank k. Then, using the above notation,*

(i) $\Bbbk[\mathfrak{k}^*]^{K^u}$ *is a polynomial algebra that is freely generated by the coordinates on $\mathfrak{g}_1^*$ and $\widehat{F}_j$, $j = k+1, \dots, l$;*
(ii) $\Bbbk[\mathfrak{k}^*]^K$ *is the polynomial algebra that is freely generated by $\bar{f}_1, \dots, \bar{f}_k, \widehat{F}_{k+1}, \dots, \widehat{F}_l$.*

Proof. It follows from Eq. (5.1) that $f_i^\bullet = \bar{f}_i$ for $i \leqslant k$ and $f_j^\bullet = \widehat{F}_j$ for $j \leqslant k+1$. Regarding all these functions as functions on $\mathfrak{k}^*$, we obtain by virtue of Proposition 3.1 that $\bar{f}_1, \dots, \bar{f}_k, \widehat{F}_{k+1}, \dots, \widehat{F}_l$ belong to $\Bbbk[\mathfrak{k}^*]^K$.

The proof below is quite similar to that of Theorem 6.2 in [13]. To prove part (i), we use Igusa's lemma (see [23, Theorem 4.12] or [13, Lemma 6.1]) and properties of the covariants F_i in Eq. (5.1). Part (ii) is then an obvious consequence of (i).

For (i): Let $\Omega \subset \mathfrak{g}_1^*$ be the open subset of G_0-regular elements. As follows from [8], Ω is big and any $\zeta \in \Omega$ is also regular as element of $\mathfrak{g}$. The functions indicated in (i) are clearly K^u-invariant and are algebraically independent (consider their differentials at some $\xi \in \Omega$). Hence we obtain the dominant mapping

$$\psi : \mathfrak{k}^* \to \mathfrak{g}_1^* \times \Bbbk^{l-k},$$

defined by $\psi(\xi_0, \xi_1) = (\xi_1, \widehat{F}_{k+1}(\xi_0, \xi_1), \dots, \widehat{F}_l(\xi_0, \xi_1))$. Since the vectors $F_j(\zeta)$, $j = k+1, \dots, l$, form a basis of $\mathfrak{g}_{0,\zeta}$ for any $\zeta \in \Omega$, we see that $\Omega \times \Bbbk^{l-k} \subset \operatorname{Im}\psi$, i.e., $\operatorname{Im}\psi$ contains a big open subset of $\mathfrak{g}_1^* \times \Bbbk^{l-k}$. If $(\zeta, z_{k+1}, \dots, z_l) \subset \Omega \times \Bbbk^{l-k}$, then

$$\psi^{-1}(\zeta, z_{k+1}, \dots, z_l) = \bigl\{(\xi_0, \zeta) \bigm| \langle \xi_0, F_j(\zeta)\rangle = z_j,\ j \geqslant k+1\bigr\}.$$

It is a K^u-stable affine subspace of $\mathfrak{k}^*$ of dimension $\dim\mathfrak{g}_0 - (l-k)$. On the other hand, if $(\xi_0, \zeta) \in \psi^{-1}(\zeta, z_{k+1}, \dots, z_l)$, then

$$K^u \cdot (\xi_0, \zeta) = (1 \ltimes \mathfrak{g}_1) \cdot (\xi_0, \zeta) = \bigl\{(\xi_0 + x_1 * \zeta, \zeta) \bigm| x_1 \in \mathfrak{g}_1\bigr\}.$$

Upon the identification of $\mathfrak{g}_1$ and $\mathfrak{g}_1^*$, we have $\mathfrak{g}_1 * \zeta = [\mathfrak{g}_1, \zeta]$. Hence

$$\dim\bigl(K^u \cdot (\xi_0, \zeta)\bigr) = \dim(\mathfrak{g}_1 * \zeta) = \dim\mathfrak{g}_1 - k = \dim\mathfrak{g}_0 - (l-k).$$

Since the orbits of unipotent groups on affine varieties are closed [17, Theorem 2] and isomorphic to affine spaces, we conclude that $\psi^{-1}(\zeta, z_{k+1}, \dots, z_l) = K^u(\xi_0, \zeta)$, i.e., almost all fibres of ψ are precisely K^u-orbits.

Hence all the assumptions of Igusa's lemma are satisfied, and part (i) follows.

A direct proof for part (ii) (without using (i)) is as follows. The K-invariants $\bar f_1, \dots, \bar f_k$, $\widehat{F}_{k+1}, \dots, \widehat{F}_l$ are $\mathbb{Z}_2$-degenerations of $f_1, \dots, f_l$, hence they have the same degrees. It is also easily seen that these K-invariants are algebraically independent. Next, we know that $\operatorname{ind}\mathfrak{k} = \operatorname{ind}\mathfrak{g}$. Therefore, Theorem 1.2(ii) applies in this situation. $\square$

Remark. If $\operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0) = l = \operatorname{rk}\mathfrak{g}$, then the above theorem merely says that $\Bbbk[\mathfrak{k}^*]^{K^u} \simeq \Bbbk[\mathfrak{g}_1]$ and $\Bbbk[\mathfrak{k}^*]^K \simeq \Bbbk[\mathfrak{g}_1]^{G_0}$. This was already observed, in a more general context, in [13, Theorem 6.4]. So, the novelty of Theorem 5.2 concerns the case in which $\operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0) < \operatorname{rk}\mathfrak{g}$.

5.3. Theorem. *If $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$ is the $\mathbb{Z}_2$-contraction of an $\mathcal{N}$-regular $\mathbb{Z}_2$-grading, then $\pi_{\mathfrak{k}^*} : \mathfrak{k}^* \to \mathfrak{k}^* /\!\!/ K$ is equidimensional.*

Proof. Keep the notation of the previous proof. If $\operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0) = \operatorname{rk}\mathfrak{g}$, then the isomorphism $\Bbbk[\mathfrak{k}^*]^K \simeq \Bbbk[\mathfrak{g}_1]^{G_0}$ shows that $\mathcal{N}(\mathfrak{k}^*) \simeq \mathcal{N}(\mathfrak{g}_1) \times \mathfrak{g}_0$. Hence the assertion.

Assume therefore that $k = \operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0) < \operatorname{rk}\mathfrak{g} = l$ and hence there are non-trivial K-invariants of the form $\widehat{F}_j$. Roughly speaking, the covariants $F_j \in \operatorname{Mor}_{G_0}(\mathfrak{g}_1, \mathfrak{g}_0)$, $j = k+1, \dots, l$, determine a stratification of the null-cone $\mathcal{N}(\mathfrak{g}_1)$, and the assertion is equivalent to certain property of this stratification. Unfortunately, we can only verify that property using a case-by-case argument. (This is very similar to our proofs for $(\mathfrak{k}, \mathrm{ad})$ in [13, Section 9].)

Set $\mathcal{N}(\mathfrak{k}^*) = \pi_{\mathfrak{k}^*}^{-1}(\pi_{\mathfrak{k}^*}(0))$. Then $\operatorname{codim}\mathcal{N}(\mathfrak{k}^*) \leqslant l$ and the equidimensionality of $\pi_{\mathfrak{k}^*}$ precisely means that $\operatorname{codim}\mathcal{N}(\mathfrak{k}^*) = l$. If $(\alpha, \xi) \in \mathcal{N}(\mathfrak{k}^*)$, then the inclusion $\Bbbk[\mathfrak{g}_1]^{G_0} \hookrightarrow \Bbbk[\mathfrak{k}^*]^K$ shows that $\xi \in \mathcal{N}(\mathfrak{g}_1)$. Hence we obtain the surjective projection $p : \mathcal{N}(\mathfrak{k}^*) \to \mathcal{N}(\mathfrak{g}_1)$, $(\alpha, \xi) \mapsto \xi$. Let J denote the (finite) set of G_0-orbits in $\mathcal{N}(\mathfrak{g}_1)$. Then

$$\mathcal{N}(\mathfrak{k}^*) = \bigsqcup_{\mathcal{O} \in J} p^{-1}(\mathcal{O})$$

and the irreducible components are contained among the sets $\overline{p^{-1}(\mathcal{O})}$. Hence the assertion is equivalent to the condition that $\dim p^{-1}(\mathcal{O}) \leqslant \dim\mathfrak{k}^* - l$ for all $\mathcal{O} \in J$. Since $\dim p^{-1}(\xi) = \dim\mathfrak{g}_0 - \dim\operatorname{span}\{F_{k+1}(\xi), \dots, F_l(\xi)\}$ for $\xi \in \mathcal{N}(\mathfrak{g}_1)$, our condition readily translates as follows: For any $\xi \in \mathcal{N}(\mathfrak{g}_1)$, we should have

$$l \leqslant \dim\mathfrak{g}_{1,\xi} + \dim\operatorname{span}\bigl\{F_{k+1}(\xi), \dots, F_l(\xi)\bigr\}. \tag{5.4}$$

Recall that $\dim\mathfrak{g}_{1,\xi} \geqslant k = \operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0)$ and $\dim\mathfrak{g}_{1,\xi} = k$ if and only if $\xi \in \Omega$.

The list of $\mathcal{N}$-regular symmetric pairs such that $\mathfrak{g}$ is simple and $\operatorname{rk}\mathfrak{g} > \operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0)$ is given below:

- $(\mathfrak{sl}_{n+m}, \mathfrak{sl}_n \dotplus \mathfrak{sl}_m \dotplus \Bbbk)$ with $|n - m| \leqslant 1$;
- $(\mathfrak{so}_{2n+2}, \mathfrak{so}_n \dotplus \mathfrak{so}_{n+2})$;
- $(\mathbf{E}_6, \mathfrak{sl}_6 \dotplus \mathfrak{sl}_2)$.

In the second case, $l = n + 1$ and $k = n$. Here there is only one covariant, F_l, and the equidimensionality is obvious.

In the third case, $l - k = 6 - 4 = 2$, and there are two covariants F_5, F_6. Here Eq. (5.4) essentially means that if $\xi \in \mathcal{N}(\mathfrak{g}_1)$ and $\operatorname{codim}_{\mathcal{N}(\mathfrak{g}_1)}(G_0 \cdot \xi) = 1$, then at least one of the covariants F_5, F_6 does not vanish at ξ. To this end, we notice that $G \cdot \xi$ is the subregular nilpotent orbit, $\mathcal{O}_{\mathrm{sub}}$, For $\xi \in \mathcal{O}_{\mathrm{sub}}$, we have $\dim\operatorname{span}\{(df_1)_\xi, \dots, (df_l)_\xi\} = l - 1$ [3]. For our "adapted" choice of basic invariants $f_1, \dots, f_l$, as above, we have $(df_j)_\xi = F_j(\xi)$ for $j \geqslant k+1$ and $\xi \in \mathfrak{g}_1$. Hence two covariants F_j cannot vanish on $\mathcal{O}_{\rm sub} \cap \mathfrak{g}_1$, which is exactly what we need.

For the first case, Eq. (5.4) will be verified in Example 5.6 below. $\square$

The following is a standard consequence of Theorems 5.2(ii) and 5.3.

5.5. Corollary. *Let $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$ be the contraction of an $\mathcal{N}$-regular $\mathbb{Z}_2$-grading. Let $\mathcal{U}(\mathfrak{k})$ denote the enveloping algebra of $\mathfrak{k}$ and $\mathcal{Z}(\mathfrak{k})$ the centre of $\mathcal{U}(\mathfrak{k})$. Then $\mathcal{Z}(\mathfrak{k})$ is a polynomial algebra and $\mathcal{U}(\mathfrak{k})$ is a free module over $\mathcal{Z}(\mathfrak{k})$.*

5.6. Example. To simplify exposition, we work with $\mathfrak{gl}_n$ in place of $\mathfrak{sl}_n$. Let

$$\mathfrak{g} = \mathfrak{gl}_{2n} \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{gl}_n \dotplus \mathfrak{gl}_n = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Then

$$\mathfrak{g}_1 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

As always, we identify $\mathfrak{g}_i$ and $\mathfrak{g}_i^*$, $i = 0, 1$. Here $\operatorname{rk} \mathfrak{g} = 2n$ and $\operatorname{rk}(\mathfrak{g}, \mathfrak{g}_0) = n$. Set

$$\xi_0 = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \in \mathfrak{g}_0 \quad \text{and} \quad \xi_1 = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{g}_1.$$

To obtain a regular nilpotent element in $\mathfrak{g}_1$, one may take $B = I_n$ and A to be any nilpotent $n \times n$ matrix such that $A^{n-1} \neq 0$. The algebra $\Bbbk[\mathfrak{g}_1]^{G_0}$ is freely generated by the polynomials $f_i(\xi_1) = \operatorname{tr}((\xi_1)^{2i}) = \operatorname{tr}((AB)^i + (BA)^i)$, $i = 1, 2, \dots, n$. These polynomials are naturally regarded as polynomials on the whole of $\mathfrak{k}^*$. Define the covariants $F_i : \mathfrak{g}_1 \to \mathfrak{g}_0$ by the formula

$$F_i(\xi_1) = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{2i-2} = \begin{pmatrix} (AB)^{i-1} & 0 \\ 0 & (BA)^{i-1} \end{pmatrix}.$$

Obviously, $F_i(\xi_1)$ commutes with ξ_1, i.e., $F_i(\xi_1) \in \mathfrak{g}_{0,\xi_1}$. Therefore $\widehat{F}_i(\xi_0, \xi_1) := \langle \xi_0, F_i(\xi_1) \rangle = \operatorname{tr}(\xi_0(\xi_1)^{2i})$ is a K-invariant polynomial on $\mathfrak{k}^*$. If $\xi_1 \in \mathfrak{g}_1$ is a regular nilpotent element, then $\xi_1^0, \xi_1^2, \dots, \xi_1^{2n-2}$ are linearly independent. Hence $F_1, \dots, F_n$ form a basis of the $\Bbbk[\mathfrak{g}_1]^{G_0}$-module $\operatorname{Mor}_{G_0}(\mathfrak{g}_1, \mathfrak{g}_0)$. It follows that $\Bbbk[\mathfrak{k}^*]^K$ is freely generated by the polynomials

$$f_i(\xi_0, \xi_1) = \operatorname{tr}\big((AB)^i + (BA)^i\big), \quad i = 1, 2, \dots, n, \quad \text{and}$$
$$\widehat{F}_i(\xi_0, \xi_1) = \operatorname{tr}\big(M(AB)^i + N(BA)^i\big), \quad i = 1, \dots, n.$$

In this case, Eq. (5.4) for $\xi \in \mathcal{N}(\mathfrak{g}_1)$ reads

$$\dim \mathfrak{g}_{1,\xi} + \dim \operatorname{span}\big\{\xi^{2i} \bigm| i = 0, 1, \dots, n-1\big\} - 2n \geqslant 0.$$

Eliminating $i=0$ and taking into account that here $\dim \mathfrak{g}_{1,\xi} = \frac{1}{2}\dim \mathfrak{g}_\xi$, we rewrite it as

$$\frac{1}{2}\dim \mathfrak{g}_\xi + \dim \operatorname{span}\{\xi^{2i} \mid i = 1,\dots,n-1\} - 2n + 1 \geqslant 0.$$

Let $(\eta_1, \eta_2, \dots)$ be the partition of $2n$ corresponding to ξ. Then $\xi^{2i} \neq 0$ if and only if $2i \leqslant \eta_1 - 1$. Write $(\hat\eta_1, \hat\eta_2, \dots, \hat\eta_s)$ for the dual partition. This means in particular that $s = \eta_1$. It is well known that $\dim \mathfrak{g}_\xi = \sum_{i=1}^s \hat\eta_i^2$. Hence the left-hand side equals

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^s \hat\eta_i^2 + \left\lfloor \frac{\eta_1 - 1}{2} \right\rfloor - 2n + 1 &= \frac{1}{2}\sum_{i=1}^s \hat\eta_i^2 + \left\lfloor \frac{s-1}{2} \right\rfloor - \left(\sum_{i=1}^s \hat\eta_i\right) + 1 \\ &= \frac{1}{2}\left(\sum_{i=1}^s (\hat\eta_i - 1)^2 - s + 2 + 2\left\lfloor \frac{s-1}{2} \right\rfloor\right) \\ &= \frac{1}{2}\sum_{i=1}^s (\hat\eta_i - 1)^2 + \left(\left\lfloor \frac{s+1}{2} \right\rfloor - \frac{s}{2}\right),\end{aligned}$$

which is non-negative, as required.

The case of $\mathfrak{g} = \mathfrak{gl}_{2n+1}$ and $\mathfrak{g}_0 = \mathfrak{gl}_n \dotplus \mathfrak{gl}_{n+1}$ is quite similar and left to the reader.

5.7. Remarks. (1) Using a more involved analysis, we can prove that, for all $\mathcal{N}$-regular $\mathbb{Z}_2$-gradings, the ideal generated by the basic K-invariants $\bar f_1, \dots, \bar f_k, \widehat{F}_{k+1}, \dots, \widehat{F}_l$ is equal to its radical. To this end, it suffices to demonstrate that each irreducible component of $\mathcal{N}(\mathfrak{k}^*)$ contains a K-regular point.

(2) The null-fibre $\mathcal{N}(\mathfrak{k}^*)$ is often reducible. The projection $p : \mathcal{N}(\mathfrak{k}^*) \to \mathcal{N}(\mathfrak{g}_1)$ considered in Theorem 5.3 shows that $\#\mathrm{Irr}(\mathcal{N}(\mathfrak{k}^*)) \geqslant \#\mathrm{Irr}(\mathcal{N}(\mathfrak{g}_1))$, where $\#\mathrm{Irr}(\cdot)$ refers to the number of irreducible components. The numbers $\#\mathrm{Irr}(\mathcal{N}(\mathfrak{g}_1))$ are found by Sekiguchi for all symmetric pairs [18, Theorem 1]. It may happen that $\#\mathrm{Irr}(\mathcal{N}(\mathfrak{k}^*)) > \#\mathrm{Irr}(\mathcal{N}(\mathfrak{g}_1))$. For instance, if $\mathfrak{g} = \mathfrak{gl}_{2n+1}$ and $\mathfrak{g}_0 = \mathfrak{gl}_n \dotplus \mathfrak{gl}_{n+1}$, then $\mathcal{N}(\mathfrak{g}_1)$ is irreducible, while our computation shows that $\#\mathrm{Irr}(\mathcal{N}(\mathfrak{k}^*)) = 2$. The additional irreducible component appears as the closure of $p^{-1}(\mathcal{O}_{\mathrm{sub}} \cap \mathfrak{g}_1)$.

The covariants $F_{k+1}, \dots, F_l$ have another natural description. Let $\mathrm{Mor}(\mathfrak{g}_1, \mathfrak{g}_0)$ (respectively $\mathrm{Mor}(\mathfrak{g}_1, \mathfrak{g}_1)$) be the set of *all* polynomial morphisms $\mathfrak{g}_1 \to \mathfrak{g}_0$ (respectively $\mathfrak{g}_1 \to \mathfrak{g}_1$). These are free $\Bbbk[\mathfrak{g}_1]$-modules of rank $\dim \mathfrak{g}_0$ and $\dim \mathfrak{g}_1$, respectively. Consider the homomorphism $\hat\phi : \mathrm{Mor}(\mathfrak{g}_1, \mathfrak{g}_0) \to \mathrm{Mor}(\mathfrak{g}_1, \mathfrak{g}_1)$ defined by $\hat\phi(F)(x_1) = [F(x_1), x_1]$. Then $\ker\hat\phi = \{F \mid F(x_1) \in \mathfrak{g}_{0,x_1}\}$. Notice that F is not supposed to be G_0-equivariant. The homomorphism $\hat\phi$ can be defined for any symmetric pair. But the following is only true in the $\mathcal{N}$-regular case (cf. [13, Theorem 8.6]).

5.8. Theorem. *Let $(\mathfrak{g}, \mathfrak{g}_0)$ be an $\mathcal{N}$-regular symmetric pair. Then $\ker\hat\phi$ is a free $\Bbbk[\mathfrak{g}_1]$-module, and the G_0-equivariant morphisms $F_{k+1}, \dots, F_l$ form a basis of $\ker\hat\phi$.*

Proof. Clearly, $\ker\hat\phi$ is a torsion-free $\Bbbk[\mathfrak{g}_1]$-module and its rank, $\mathrm{rk}(\ker\hat\phi)$, is well defined. By definition,

$$\mathrm{rk}(\ker\hat\phi) := \dim\bigl(\ker\hat\phi \otimes_{\Bbbk[\mathfrak{g}_1]} \Bbbk(\mathfrak{g}_1)\bigr).$$

In the coordinate form, $\hat{\phi}$ is represented via a $\dim\mathfrak{g}_0 \times \dim\mathfrak{g}_1$-matrix with entries in $\Bbbk[\mathfrak{g}_1]$, and $\mathrm{rk}(\hat{\phi})$ is the rank of this matrix. Then $\mathrm{rk}(\ker\hat{\psi}) = \dim\mathfrak{g}_0 - \mathrm{rk}(\hat{\psi})$. Because

$$\mathrm{rk}\,\hat{\phi} = \max_{x\in\mathfrak{g}_1} \dim(G_0\cdot x) = \dim\mathfrak{g}_1 - k = \dim\mathfrak{g}_0 - (l-k),$$

we have $\mathrm{rk}(\ker\hat{\psi}) = l-k$. Recall that $F_{k+1}(\xi),\dots,F_l(\xi)$ are linearly independent over $\Bbbk$ for any $\xi\in\Omega$, hence $F_{k+1},\dots,F_l$ are linearly independent over $\Bbbk[\mathfrak{g}_1]$. As was noticed before, $F_{k+1},\dots,F_l\in\ker\hat{\phi}$. Hence $F_{k+1},\dots,F_l$ generate $\ker\hat{\phi}\otimes_{\Bbbk[\mathfrak{g}_1]}\Bbbk(\mathfrak{g}_1)$. That is, for any $F\in\ker\hat{\psi}$ there exist $\hat{p}, p_{k+1},\dots,p_l\in\Bbbk[\mathfrak{g}_1]$ such that

$$\hat{p}F = \sum_{i\geqslant k+1} p_i F_i.$$

Assume $\hat{p}\notin\Bbbk^*$. Let p be a prime factor of $\hat{p}$ and D the divisor of zeros of p. Then $\sum_i p_i(v)F_i(v) = 0$ for any $v\in D$. Since $\Omega\subset\mathfrak{g}_1$ is big, $\Omega\cap D$ is dense in D. Because $\{F_i(v)\}$ are linearly independent for any $v\in\Omega$, we obtain $p_i|_D\equiv 0$. Hence $p_i/p\in\Bbbk[\mathfrak{g}]$ for each i, and we are done. □

Note that $\ker\hat{\phi}$ cannot be generated by G_0-equivariant morphisms, unless $(\mathfrak{g},\mathfrak{g}_0)$ is $\mathcal{N}$-regular. The reason is that in general $\mathrm{rk}(\ker\hat{\phi}) = \dim\mathfrak{s}$, whereas one can show that the set of G_0-equivariant morphisms in $\ker\hat{\phi}$ has the rank $\dim\mathfrak{z}(\mathfrak{s})$ as the $\Bbbk[\mathfrak{g}_1]^{G_0}$-module. It remains to observe that $\mathcal{N}$-regularity precisely means that $\mathfrak{s}$ is commutative, i.e., $\mathfrak{s} = \mathfrak{z}(\mathfrak{s})$.

6. Tables

In this section, we gather the available information about the structure of algebras $\Bbbk[\mathfrak{k}]^K$ and $\Bbbk[\mathfrak{k}^*]^K$, where $\mathfrak{k} = \mathfrak{g}_0\ltimes\mathfrak{g}_1$. The case of the adjoint representation is fully covered by results of [13]. In particular, if $\mathrm{rk}\,\mathfrak{g} = \mathrm{rk}\,\mathfrak{g}_0$, i.e., the involution σ is inner, then $\Bbbk[\mathfrak{k}]^K \simeq \Bbbk[\mathfrak{g}_0]^{G_0}$. Therefore we do not always write explicitly down the respective bi-degrees. For $\Bbbk[\mathfrak{k}^*]^K$, the answer is

Table 1
Classical Lie algebras (inner involutions)

(G, G_0)	Bi-degrees for $(\mathfrak{k}, \mathrm{ad})$	Bi-degrees for $(\mathfrak{k}, \mathrm{ad}^*)$	
$(GL_{n+m}, GL_n\times GL_m)$ $n\geqslant m$	degrees of $\mathfrak{g}_0$	$(0,2),(0,4),\dots,(0,2m)$ $(1,0),(1,2),(1,4),\dots,(1,2m)$ $(2,2m),(3,2m),\dots,(n-m,2m)$	(N_1) if $n-m\leqslant 1$
$(Sp_{2n+2m}, Sp_{2n}\times Sp_{2m})$ $n\geqslant m$	degrees of $\mathfrak{g}_0$	$(0,2),(0,4),\dots,(0,2m)$ $(2,2m),(2,2m+2),\dots,(2,4m-2)$ $(2,4m),(4,4m),\dots,(2n-2m,4m)$	
$(SO_{2n+1}, SO_{n+1}\times SO_n)$	degrees of $\mathfrak{g}_0$	$(0,2),(0,4),\dots,(0,2n)$	max
$(SO_{n+m}, SO_n\times SO_m)$ $n>m+2$, $n+m$ is odd	degrees of $\mathfrak{g}_0$	$(0,2),(0,4),\dots,(0,2m)$ $(2,2m),(4,2m),\dots,(n-m-1,2m)$	
(SO_{4n}, GL_{2n})	degrees of $\mathfrak{g}_0$	$(0,2),(0,4),\dots,(0,2n)$ $(2,2n-2),(2,2n),\dots,(2,4n-4)$	
(SO_{4n+2}, GL_{2n+1})	degrees of $\mathfrak{g}_0$	$(0,2),(0,4),\dots,(0,2n),(1,2n)$ $(2,2n),(2,2n+2),\dots,(2,4n-2)$	

Table 2
Classical Lie algebras (outer involutions)

(G, G_0)	Bi-degrees for $(\mathfrak{k}, \mathrm{ad})$	Bi-degrees for $(\mathfrak{k}, \mathrm{ad}^*)$	
(SL_{2n}, SO_{2n})	$(2,0), (4,0), \dots, (2n-2,0), (n,0)$ $(2,1), (4,1), \dots, (2n-2,1)$	$(0,2), (0,3), \dots, (0,2n)$	max
(SL_{2n+1}, SO_{2n+1})	$(2,0), (4,0), \dots, (2n-2,0), (2n,0)$ $(2,1), (4,1), \dots, (2n-2,1), (2n,1)$	$(0,2), (0,3), \dots, (0,2n+1)$	max &(N_0)
(SL_{2n}, Sp_{2n})	$(2,0), (4,0), \dots, (2n-2,0), (2n,0)$ $(2,1), (4,1), \dots, (2n-2,1)$	$(0,2), (0,3), \dots, (0,n)$ *$(2,n-1), (2,n), \dots, (2,2n-2)$*	(N_0)
$(\mathbf{D}_{n+m+1}, \mathbf{B}_n \times \mathbf{B}_m)$ $n \geqslant m+2$	$(2,0), (4,0), \dots, (2n-2,0), (2n,0)$ $(2,0), (4,0), \dots, (2m-2,0), (2m,0)$ $(n+m,1)$	$(0,2), (0,4), \dots, (0,4m+2)$ $(2,4m+2), (4,4m+2), \dots,$ $(2n-2m-2,4m+2), (n-m,2m+1)$	(N_0) if $m=0$
$(\mathbf{D}_{2n}, \mathbf{B}_n \times \mathbf{B}_{n-1})$ $(n=m+1)$	$(2,0), (4,0), \dots, (2n-2,0), (2n,0)$ $(2,0), (4,0), \dots, (2m-2,0), (2m,0)$ $(2n-1,1)$	$(0,2), (0,4), \dots, (0,4n-2)$ $(1,2n-1)$	(N_1)
$(\mathbf{D}_{2n+1}, \mathbf{B}_n \times \mathbf{B}_n)$ $(n=m)$	$(2,0), (4,0), \dots, (2n-2,0), (2n,0)$ $(2,0), (4,0), \dots, (2n-2,0), (2n,0)$ $(2n,1)$	$(0,2), (0,4), \dots, (0,4n), (0,2n+1)$	max

Table 3
Exceptional Lie algebras

(G, G_0)	Bi-degrees for $(\mathfrak{k}, \mathrm{ad})$	Bi-degrees for $(\mathfrak{k}, \mathrm{ad}^*)$	
$(\mathbf{F}_4, \mathbf{B}_4)$	$(2,0), (4,0), (6,0), (8,0)$	$(0,2), (2,4), (4,4), (6,6)$	
$(\mathbf{F}_4, \mathbf{C}_3 \times \mathbf{A}_1)$	$(2,0), (2,0), (4,0), (6,0)$	$(0,2), (0,6), (0,8), (0,12)$	max
$(\mathbf{E}_6, \mathbf{C}_4)$	$(2,0), (4,0), (4,1), (6,0), (8,0), (8,1)$	$(0,2), (0,5), (0,6), (0,8), (0,9), (0,12)$	max
$(\mathbf{E}_6, \mathbf{F}_4)$	$(2,0), (4,1), (6,0), (8,0), (8,1), (12,0)$	$(0,2), (0,3)$, *$(2,?), (4,?), (4,?), (6,?)$*	(N_0)
$(\mathbf{E}_6, \mathbf{A}_5 \times \mathbf{A}_1)$	$(2,0), (2,0), (3,0), (4,0), (5,0), (6,0)$	$(0,2), (0,6), (0,8), (0,12), (1,4), (1,8)$	(N_1)
$(\mathbf{E}_6, \mathbf{D}_5 \times \mathbf{T}_1)$	$(1,0), (2,0), (4,0), (6,0), (8,0), (5,0)$	$(0,2), (0,4)$, *$(1,?), (2,?), (3,?), (4,?)$*	
$(\mathbf{E}_7, \mathbf{D}_6 \times \mathbf{A}_1)$	$(2,0), (4,0), (6,0), (8,0), (10,0), (6,0)$ $(2,0)$	$(0,2), (0,6), (0,8), (0,12),$ *$(2,8), (2,12), (2,16)$*	
$(\mathbf{E}_7, \mathbf{E}_6 \times \mathbf{T}_1)$	$(1,0), (2,0), (5,0), (6,0), (8,0), (9,0)$ $(12,0)$	$(0,2), (0,4), (0,6),$ *$(2,?), (4,?), (4,?), (6,?)$*	
$(\mathbf{E}_7, \mathbf{A}_7)$	degrees of $\mathbf{A}_7$	degrees of $\mathbf{E}_7$	max
$(\mathbf{E}_8, \mathbf{E}_7 \times \mathbf{A}_1)$	$(2,0), (6,0), (8,0), (10,0), (12,0)$ $(14,0), (18,0), (2,0)$	$(0,2), (0,6), (0,8), (0,12),$ *$(2,?), (4,?), (4,?), (6,?)$*	
$(\mathbf{E}_8, \mathbf{D}_8)$	degrees of $\mathbf{D}_8$	degrees of $\mathbf{E}_8$	max

known for the symmetric pairs considered in Sections 4 and 5. In the maximal rank case, we have $\Bbbk[\mathfrak{k}^*]^K \simeq \Bbbk[\mathfrak{g}_1]^{G_0} \simeq \Bbbk[\mathfrak{g}]^G$, and in two such cases we omit indication of the degrees. For all other pairs not mentioned in Remark 4.3, we have precise suggestions for the degrees. These conjectural degrees are displayed in *italic*. For the four cases mentioned in Remark 4.3, we put the question mark, if there is no suggestion for the corresponding degree, see Table 3.

The last column contains a comment on the pair in question: "max" means the maximal rank case; (N_i) means that $\mathfrak{g}_i$ contains a regular nilpotent element of $\mathfrak{g}$, $i = 0, 1$. [Hence $(N_1) = \mathcal{N}$-regular.]

Recall that $\mathrm{rk}\,\mathfrak{g} = \mathrm{rk}(\mathfrak{g}, \mathfrak{g}_0) + \mathrm{rk}\,\mathfrak{s}$, $\dim \mathfrak{g}/\!\!/G_0 = \mathrm{rk}(\mathfrak{g}, \mathfrak{g}_0)$, and $\Bbbk[\mathfrak{g}_1]^{G_0}$ is embedded in $\Bbbk[\mathfrak{k}^*]^K$. Hence we always have $\mathrm{rk}(\mathfrak{g}, \mathfrak{g}_0)$ basic invariants whose $\mathfrak{g}_0$-degree equals 0. There is an a posteriori observation related to the $\mathfrak{g}_0$-degrees of the remaining basic invariants in $\Bbbk[\mathfrak{k}^*]^K$. Namely, they are equal to the basic degrees of $\mathfrak{s}$ in all cases, where the algebra $\Bbbk[\mathfrak{k}^*]^K$ is known. For instance, look at the first pair in Table 1. Here $\mathfrak{s} \simeq \mathfrak{gl}_{n-m} \times \mathfrak{t}_m$ and the nonzero $\mathfrak{g}_0$-degrees are 1 (m times), $1, 2, 3, \ldots, n-m$.

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